Math 220 Fall 2021, Lecture 9: Quantifiers and logic

0.1. Quantifiers (cont'd)

Last time, we introduced the all-quantifier, which gives us a way to state that something is true for all elements of a set (or all objects of some type). The syntax is " $\forall x \in S : P(x)$ ".

Let us state some common statements as formulas using this quantifier:

- "The square of a real number is at least 0." This becomes

  $$\forall x \in \mathbb{R} : x^2 \geq 0.$$

  (This is pronounced "for all x in the set $\mathbb{R}$, we have $x^2 \geq 0$".)

- "The cube function is strictly increasing" (aka "the greater the number, the greater its cube"). This becomes

  $$\forall x, y \in \mathbb{R} : \left(x < y \implies x^3 < y^3 \right).$$

  (This is pronounced "for all x and y in the set $\mathbb{R}$, we have the following: If $x < y$, then $x^3 < y^3$". You can shorten this to "if two real numbers x and y satisfy $x < y$, then $x^3 < y^3$".)

  Instead of "$\forall x, y \in \mathbb{R}$", you could also say "$\forall x \in \mathbb{R} : \forall y \in \mathbb{R}$".

- "The sine function is strictly increasing" (a false proposition). This becomes

  $$\forall x, y \in \mathbb{R} : (x < y \implies \sin x < \sin y).$$

  This is false, because e.g., we have $0 < \pi$ but we don't have $\sin 0 < \sin \pi$.

- "The sine function is strictly increasing on the interval $[-\pi/2, \ \pi/2]$" (a true proposition). This becomes

  $$\forall x, y \in \mathbb{R} : (-\pi/2 \leq x < y \leq \pi/2 \implies \sin x < \sin y)$$

  or, alternatively,

  $$\forall x, y \in [-\pi/2, \pi/2] : (x < y \implies \sin x < \sin y)$$

- "The square of an odd integer is odd." This becomes

  $$\forall m \in \mathbb{Z} : \left(m \text{ is odd} \implies m^2 \text{ is odd} \right).$$

- "The three altitudes of a triangle have a point in common." To formalize this, we assume that "triangle", "point", "line" and "perpendicular" are well-defined, and there is a notation XY for the line connecting two points X and Y, and a notation $\operatorname{perp}(P, \ell)$ for the perpendicular from a point P to a line ℓ. Then, the statement becomes

$$\forall ABC \text{ triangle} : \operatorname{perp}(A, BC) \cap \operatorname{perp}(B, CA) \cap \operatorname{perp}(C, AB) \neq \varnothing.$$

So much for the all-quantifier. There is another quantifier:

Definition 0.1.1. Let S be a set. Let $P(x)$ be a predicate depending on a variable x that belongs to the set S (for example, "x is even" when $S = \mathbb{Z}$, or "x has 3 elements" when $S = \{\text{sets}\}$). Then, the statement "$P(x)$ holds for **at least one** $x \in S$" (aka "there exists **some** $x \in S$ for which $P(x)$ holds") is written

$$\text{“ } \exists x \in S : P(x) \text{ ”.}$$

This symbol $\exists$ stands for "exists", and this kind of quantification is called **existential quantification**.

Example 0.1.2. The statement

$$\text{“ } \exists x \in \mathbb{Z} : x^2 = 5 \text{ ”}$$

means "there is an integer x such that $x^2 = 5$", or, for short, "there is an integer whose square is 5". This is often restated as "5 is a perfect square". Of course, it is a false statement.

The statement

$$\text{“ } \exists x \in \mathbb{Q} : x^2 = 5 \text{ ”}$$

means "there is a rational number x such that $x^2 = 5$". Still false.

The statement

$$\text{“ } \exists x \in \mathbb{R} : x^2 = 5 \text{ ”}$$

means "there is a real number x such that $x^2 = 5$". This one is true, because $\sqrt{5}$ is such a real number.

Let us restate some common propositions as formulas using the existential quantifier:

- "There is no real number whose square is negative". This becomes

$$\text{NOT } \left(\exists x \in \mathbb{R} : x^2 < 0 \right).$$

- "Each odd integer can be written as $2k + 1$ for some integer k". This becomes

$$\forall n \in \mathbb{Z} : (n \text{ is odd} \implies (\exists k \in \mathbb{Z} : n = 2k + 1)).$$

Reading this literally, this is saying "For each $n \in \mathbb{Z}$, the following holds: If n is odd, then there exists some $k \in \mathbb{Z}$ such that $n = 2k + 1$.".

Another way to formalize this statement would be

$$\forall n \in \{\text{odd integers}\} : (\exists k \in \mathbb{Z} : n = 2k + 1) .$$

- "Each integer that is the square of a rational number is actually the square of an integer.". This becomes

$$\forall n \in \mathbb{Z} : \left(\left(\exists x \in \mathbb{Q} : x^2 = n\right) \Longrightarrow \left(\exists y \in \mathbb{Z} : y^2 = n\right)\right) .$$

  Reading this literally, this is saying "For each $n \in \mathbb{Z}$, the following holds: If there exists an $x \in \mathbb{Q}$ such that $x^2 = n$, then there exists a $y \in \mathbb{Z}$ such that $y^2 = n$".

  By the way, we could use the same letter for x and for y, since these two statements are properly separated.

- "Any integer that is divisible by 2 and divisible by 3 is divisible by 6." This becomes

$$\forall n \in \mathbb{Z} : ((2 \mid n \text{ AND } 3 \mid n) \Longrightarrow (6 \mid n)) .$$

  (Recall that "$a \mid n$" is notation for "n is divisible by a".)

- "If any divisor of an integer n is even, then n is even." This becomes

$$\forall n \in \mathbb{Z} : ((\exists d \in \mathbb{Z} : (d \mid n \text{ AND } d \text{ is even})) \Longrightarrow (n \text{ is even})) .$$

The last two of these examples illustrate a confusing quirk of language: The word "any" can mean "all" or "exists" depending on its position in the context. So be careful with this word.

Existential statements and for-all statements can be negated. When you negate such a statement, the quantifier flips:

Theorem 0.1.3 (de Morgan's laws for quantifiers)**.** Let S be a set. Let $P(x)$ be a predicate depending on a variable x from this set S. Then:

(a) We have

$$(\text{NOT } (\forall x \in S : P(x))) \iff (\exists x \in S : (\text{NOT } P(x))) .$$

In words: To say "it is not true that every $x \in S$ satisfies $P(x)$" is tantamount to saying "there exists some $x \in S$ that satisfies NOT $P(x)$".

(b) We have

$$(\text{NOT } (\exists x \in S : P(x))) \iff (\forall x \in S : (\text{NOT } P(x))) .$$

In words: To say "it is not true that there exists an $x \in S$ satisfying $P(x)$" (usually, of course, one shortens this to "there exists no $x \in S$ satisfying $P(x)$") is tantamount to saying "for all $x \in S$, we have NOT $P(x)$".

This is fairly intuitive by considering some examples: To say that "not all Canadians speak French" is tantamount to saying that "there is some Canadian somewhere that does not speak French". On the other hand, to say that "no Canadian speaks French" is tantamount to saying that "whatever Canadian you find, he doesn't speak French".

Remark 0.1.4. De Morgan's laws for AND and OR are particular cases of de Morgan's laws for quantifiers. Indeed, if A_1 and A_2 are two statements, then

$$A_1 \text{ AND } A_2 \qquad \text{is equivalent to} \qquad \forall i \in \{1,2\} : A_i,$$

whereas

$$A_1 \text{ OR } A_2 \qquad \text{is equivalent to} \qquad \exists i \in \{1,2\} : A_i.$$

Now, there are a few more quantifiers in natural language. Mathematically, they can be reduced to $\forall$ and $\exists$ quantifiers:

- How would you say "there are at least two"? For example, "there are at least two integers x such that $x^2 = 4$" can be formalized as

  $$\exists x, y \in \mathbb{Z} : \left(x^2 = 4 \text{ AND } y^2 = 4 \text{ AND } x \neq y\right).$$

  The "AND $x \neq y$" part is important: The statement

  $$\exists x, y \in \mathbb{Z} : \left(x^2 = 4 \text{ AND } y^2 = 4\right)$$

  would mean that there exist two **possibly equal** integers whose square is 4; but this would be no better than just saying that there exists one such integer.

  Soon, when we speak about sizes of sets, we will learn a better way to formalize the above statement, namely

  $$\left|\left\{x \in \mathbb{Z} \mid x^2 = 4\right\}\right| \geq 2.$$

- How would you say "there is exactly one" (i.e., "one and only one")? For example, "there is exactly one integer x such that $x^3 = 8$" can be formalized as

  $$\left(\exists x \in \mathbb{Z} : x^3 = 8\right) \text{ AND } \left(\forall x, y \in \mathbb{Z} : \left(\left(x^3 = 8 \text{ AND } y^3 = 8\right) \Longrightarrow x = y\right)\right).$$

  This can be read as "there exists an $x \in \mathbb{Z}$ satisfying $x^3 = 8$, and furthermore, if x and y are two integers such that $x^3 = 8$ and $y^3 = 8$, then $x = y$".

  Again, using sizes of sets, this can be simplified to

  $$\left|\left\{x \in \mathbb{Z} \mid x^3 = 8\right\}\right| = 1.$$

  Also, some people use a special quantifier "$\exists!$" for "there is exactly one".