Math 220 Fall 2021, Lecture 8: Quantifiers

0.1. Sets (cont'd)

Definition 0.1.1. Two sets S and T are said to be **disjoint** if they have no element in common, i.e., if $S \cap T = \varnothing$.

For example, the sets $\{2,4\}$ and $\{1,7\}$ are disjoint. But the sets $\{2,4\}$ and $\{2,7\}$ are not.

We notice that our notions of union and intersection can be extended to multiple sets:

Definition 0.1.2. Let $S_1, S_2, \ldots, S_k$ be k sets. Then,

$$\begin{aligned} S_1 \cup S_2 \cup \cdots \cup S_k &= \{x \mid x \text{ lies in some } S_i\} \\ &= \{x \mid x \in S_1 \text{ OR } x \in S_2 \text{ OR } \cdots \text{ OR } x \in S_k\} \\ &= ((((S_1 \cup S_2) \cup S_3) \cup S_4) \cup \cdots) \cup S_k. \end{aligned}$$

When $k = 0$, this is the empty set.

Definition 0.1.3. Let $S_1, S_2, \ldots, S_k$ be k sets with $k > 0$. Then,

$$\begin{aligned} S_1 \cap S_2 \cap \cdots \cap S_k &= \{x \mid x \text{ lies in all } S_i\} \\ &= \{x \mid x \in S_1 \text{ AND } x \in S_2 \text{ AND } \cdots \text{ AND } x \in S_k\} \\ &= ((((S_1 \cap S_2) \cap S_3) \cap S_4) \cap \cdots) \cap S_k. \end{aligned}$$

This makes no sense for $k = 0$.

Here is another important way of combining sets:

Definition 0.1.4. Let S and T be two sets. Then, we define $S \times T$ to be set of all ordered pairs (s, t), where $s \in S$ and $t \in T$.

Okay, but what is an "ordered pair"?

An **ordered pair** (short: **pair**) is a list consisting of two objects. Unlike a set, an ordered pair has a well-defined order – i.e., it has a well-defined first entry and a well-defined second entry. For example, $(2, 3)$ and $(3, 2)$ are two different ordered pairs, even though $\{2, 3\}$ and $\{3, 2\}$ are the same set. Also, the two entries in an ordered pair can be equal.

The set $S \times T$ is called the **Cartesian product** (or just **product**) of S and T.

Example 0.1.5. We have

$$\begin{aligned} \{1,2\} \times \{7,8,9\} &= \{(s,t) \mid s \in \{1,2\} \text{ and } t \in \{7,8,9\}\} \\ &= \{(1,7),\ (1,8),\ (1,9),\ (2,7),\ (2,8),\ (2,9)\} \end{aligned}$$

and

$$\begin{aligned}\{1\} \times \{7,8,9\} &= \{(s,t) \mid s \in \{1\} \text{ and } t \in \{7,8,9\}\} \\ &= \{(1,7), (1,8), (1,9)\}\end{aligned}$$

and

$$\begin{aligned}\varnothing \times \{7,8,9\} &= \{(s,t) \mid s \in \varnothing \text{ and } t \in \{7,8,9\}\} \\ &= \varnothing \qquad \text{(since there exists no } s \in \varnothing\text{)}\end{aligned}$$

and

$$\{1,2\} \times \{1,2\} = \{(1,1), (1,2), (2,1), (2,2)\}.$$

Claim: The set

$$\mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\} = \{\text{pairs of two real numbers}\}$$

is a model for the plane, called the **Cartesian plane**. In what sense? In the sense that a point in a given plane can be specified by a pair of two real numbers (viz., its Cartesian coordinates with respect to a fixed coordinate system), and thus we can think of the plane as the set of all pairs of real numbers.

The notion of a Cartesian product can, too, be generalized to k sets:

Definition 0.1.6. Let $S_1, S_2, \ldots, S_k$ be k sets. Then, the **Cartesian product** $S_1 \times S_2 \times \cdots \times S_k$ is defined to be the set of all k-tuples $(s_1, s_2, \ldots, s_k)$ with $s_1 \in S_1$ and $s_2 \in S_2$ and $\cdots$ and $s_k \in S_k$.

A k**-tuple** just means a list consisting of k objects, with a well-defined order (i.e., it has a first entry, a second entry, and so on, and a k-th entry). (Again, the objects don't have to be distinct.)

Example 0.1.7. We have

$$\begin{aligned}&\{1,2\} \times \{1,2\} \times \{1,2\} \\ &= \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,2,2), (2,2,1), (2,1,2), (2,1,1)\}.\end{aligned}$$

So $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is the set of all 3-tuples (aka **triples**) of real numbers. These correspond to points in space, just as the pairs correspond to points in a plane.

Minor pedantic **warning**: The set $A \times (B \times C)$ is not the same as $A \times B \times C$ and also not the same as $(A \times B) \times C$. Indeed, the first set consists of nested pairs $(a, (b,c))$, while the second consists of triples (a,b,c), and the third consists of nested pairs $((a,b), c)$. All of these structures carry the exact same information, but they are organized differently.

Definition 0.1.8. Let S be a set, and k a nonnegative integer. Then,

$$S^k := \underbrace{S \times S \times \cdots \times S}_{k \text{ times}}.$$

This is called the **k-th (Cartesian) power** of S.

In particular, S^0 is the Cartesian product of no sets. By definition, this is a one-element set, consisting only of the empty list $()$. So it can be written

$$S^0 = \{()\}.$$

Finally, there is one more way of transforming sets:

Definition 0.1.9. Let S be a set. Then, $\mathcal{P}(S)$ denotes the set of all subsets of S. This is called the **powerset** (or **power set**) of S.

Example 0.1.10. We have

$$\mathcal{P}(\{1,2,3\}) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \varnothing\}.$$

0.2. Quantifiers

Let us return to propositions. Most of the propositions we have seen include words like "Let n be an integer" or "for each integer" or "for all positive integers". We have so far been treating these words as context. But actually, they are part of mathematical language – known as **quantifiers**. Let us explain what they mean.

Definition 0.2.1. Let S be a set. Let $P(x)$ be a predicate depending on a variable x that is supposed to belong to the set S. (For instance, "x is even" is a predicate for $S = \mathbb{Z}$. For another example, "x has 3 elements" is a predicate for $S = \{\text{sets}\}$, at least if we pretend that $\{\text{sets}\}$ is really a well-defined set.)

Then, the statement "$P(x)$ holds for all $x \in S$" (aka "for every $x \in S$, we have $P(x)$", aka "for each $x \in S$, we have $P(x)$") is formally written

$$\text{“ } \forall x \in S : P(x) \text{ ”},$$

and it is called the "**universal quantification** of $P(x)$ over the set S".

Note that $P(x)$ (taken alone) is a predicate – i.e., its meaning depends on x. However, " $\forall x \in S : P(x)$ " is a proposition; its meaning does not depend on the context for x.

Example 0.2.2. What does it mean to say

$$\text{“ } \forall x \in \mathbb{Z} : x\left(x-1\right) \geq 0 \text{ ”?}$$

It means that for each integer x, we have $x\left(x-1\right) \geq 0$. This proposition is true, by the way.

What does it mean to say

$$\text{“ } \forall x \in \mathbb{Q} : x\left(x-1\right) \geq 0 \text{ ”?}$$

It means that for each rational number x, we have $x\left(x-1\right) \geq 0$. This proposition is false (just take $x = \dfrac{1}{2}$).

Sometimes, we omit the "$\in S$" part in a universal quantification – i.e., we just write " $\forall x : P\left(x\right)$ " instead of " $\forall x \in S : P\left(x\right)$ ". As we saw in the example above, this is dangerous, unless S is clear from the context.

As we hinted above, S doesn't really need to be a set. It just needs to be some collection of things. For example, S can be the collection of all sets. This is not itself a set, but it is meaningfull to say "For all sets x", so we can pretend that we are saying " $\forall x \in \{\text{sets}\}$ ".

There are some variant notations. First of all, instead of the colon in " $\forall x \in S : P\left(x\right)$ ", some people put a comma or a period; i.e., they write " $\forall x \in S,\ P\left(x\right)$ " or " $\forall x \in S\ .\ P\left(x\right)$ ".

You can write " $\forall x$ integer" instead of " $\forall x \in \mathbb{Z}$". For instance,

$$\forall\ x \text{ set } :\ x \in \mathcal{P}\left(x\right).$$

This is simply saying that each set x satisfies $x \in \mathcal{P}\left(x\right)$, i.e., x is a subset of x. Normally, people use uppercase letters for sets, so they would write

$$\forall\ X \text{ set } :\ X \in \mathcal{P}\left(X\right)$$

instead. It doesn't matter how we call the variable, as long as the name we use for it is not simultaneously being used for something else (e.g., we cannot say "$\forall\ 2 \in \mathbb{Z}$").

We can quantify over several variables at the same time. The meaning of that is analogous to the meaning of quantifying over one variable. For instance,

$$\text{“ } \forall x, y \in \mathbb{Q} : \left(x+y\right)^2 = x^2 + 2xy + y^2 \text{ ”}$$

is saying that the equation $\left(x+y\right)^2 = x^2 + 2xy + y^2$ holds for every two rational numbers x and y.

Instead of saying "for all", we often say "for every" or "for each". (Sometimes we can also say "for any", but this is somewhat slippery; the word "any" has different meanings in different context.)