Math 220 Fall 2021, Lecture 7: More sets

0.1. Sets (cont'd)

Last time, we discussed what a set is and how sets are written.

Are $\{1,2,3\}$ and $\{2,3,1\}$ the same set? We haven't been clear about this, so let us clarify it now.

A set either contains an element, or does not. This is all that a set "knows how to do". You can think of a set as a machine that you can give an element to, and that will answer whether it contains the element. If you ask it twice whether it contains some object, it will give you the same answer both times, but that doesn't mean that it contains the object twice. Furthermore, its elements do not come with a pre-chosen order.

So $\{1,2,3\}$ and $\{2,3,1\}$ are the same set, i.e., we have $\{1,2,3\} = \{2,3,1\}$. Furthermore, $\{1,2\}$ and $\{1,1,2\}$ and $\{1,1,1,1,2,2,2,2,2,1,1,1,1\}$ are all the same set. But $\{1,2,3\}$ is not the same set as $\{1,2,1\}$.

More generally, two sets S and T are equal if they contain the same elements (i.e., if each element of S is an element of T and vice versa).

At this point, it is useful to define some more notations:

Definition 0.1.1. Let S and T be two sets.

(a) We say that S is a **subset** of T (and we write $S \subseteq T$) if each element of S is an element of T.

(b) We say that S is a **superset** of T (and we write $S \supseteq T$) if each element of T is an element of S.

Example 0.1.2. We have $\{1,3\} \subseteq \{1,2,3,4\}$.

We also have $\{1,3\} \subseteq \{1,3\}$. More generally, $S \subseteq S$ for each set S.

Is $\{1,3\}$ a subset of $\{1,2\}$? No, because it is not true that each element of $\{1,3\}$ is an element of $\{1,2\}$ (indeed, 3 is not).

We have $\{1,2,3,4\} \supseteq \{1,3\}$ and $\{1,3\} \supseteq \{1,3\}$.

Is $\{1,3\}$ a superset of $\{1,2\}$? No, because it is not true that each element of $\{1,2\}$ is an element of $\{1,3\}$ (indeed, 2 is not).

Example 0.1.3. Here is a chain of subsets:

$$\begin{aligned} &\varnothing \subseteq \{1\} \subseteq \{1,2\} \subseteq \{1,2,3\} \subseteq \{1,2,3,4\} \subseteq \cdots \subseteq \{1,2,3,4,\ldots\} \\ &\quad \subseteq \{0,1,2,3,\ldots\} = \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}. \end{aligned}$$

This chain can be continued forever, although with less mathematically important objects than $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

Each set S satisfies $\varnothing \subseteq S$, because each element of $\varnothing$ is an element of S (indeed, this is vacuously true, since there are no elements of $\varnothing$). Also, as we said, each set S satisfies $S \subseteq S$.

Theorem 0.1.4. Let S and T be two sets. We have $S \subseteq T$ if and only if $T \supseteq S$.

Recall that "if and only if" is a way of stating equivalence. So when we say that a proposition P holds "if and only if" a proposition Q holds, we mean that P and Q are equivalent, i.e., P and Q are both true or both false.

Proof of Theorem. We must prove that $(S \subseteq T) \iff (T \supseteq S)$.

To do so, we shall show that $(S \subseteq T) \implies (T \supseteq S)$ and $(T \supseteq S) \implies (S \subseteq T)$. Indeed, if P and Q are two propositions, then the proposition "$P \iff Q$" is equivalent to "$(P \implies Q)$ and $(Q \implies P)$" (you can check this using truth tables).

So let us show that $(S \subseteq T) \implies (T \supseteq S)$. If $S \subseteq T$, then each element of S is an element of T; but this means precisely that $T \supseteq S$. So $(S \subseteq T) \implies (T \supseteq S)$ is proved. Similarly, we can see that $(T \supseteq S) \implies (S \subseteq T)$ holds. This proves the Theorem. $\square$

Note that a set is always a subset of itself and also a superset of itself. If you want to exclude this, you talk about "**proper subsets**". Namely, a set S is said to be a **proper subset** of a set T if and only if $S \subseteq T$ and $S \neq T$. The notation for this is $S \subsetneq T$. Do not mistake this notation for the notation $S \not\subseteq T$, which means that S is not a subset of T. For example, it is true that $\{1,3\} \not\subseteq \{1,2\}$, but it is not true that $\{1,3\} \subsetneq \{1,2\}$.

Here are some things we can do with two sets:

Definition 0.1.5. Let S and T be two sets. Then, we define four sets

$$S \cup T, \qquad S \cap T, \qquad S \setminus T, \qquad S \bigtriangleup T$$

as follows:

- We define $S \cup T$ to be the set of all objects that belong to at least one of S and T. That is,
$$S \cup T = \{x \mid x \in S \text{ OR } x \in T\}.$$
- We define $S \cap T$ to be the set of all objects that belong to both S and T. That is,
$$S \cap T = \{x \mid x \in S \text{ AND } x \in T\}.$$
- We define $S \setminus T$ to be the set of all objects that belong to S but not to T. That is,
$$S \setminus T = \{x \mid x \in S \text{ AND } (\text{NOT } x \in T)\}.$$
- We define $S \bigtriangleup T$ to be the set of all objects that belong to exactly one of S and T. That is,
$$S \bigtriangleup T = \{x \mid x \in S \text{ XOR } x \in T\}.$$

These four sets

$$S \cup T, \qquad S \cap T, \qquad S \setminus T, \qquad S \triangle T$$

are called (respectively) the **union**, the **intersection**, the **set difference** and the **symmetric difference** of S and T.

If T is a subset of S, then $S \setminus T$ is also called the **complement** of T in S.

Example 0.1.6. We have

$$\{1,3\} \cup \{1,2\} = \{1,2,3\};$$
$$\{1,3\} \cap \{1,2\} = \{1\};$$
$$\{1,3\} \setminus \{1,2\} = \{3\};$$
$$\{1,3\} \triangle \{1,2\} = \{2,3\}.$$

Example 0.1.7. The notation $\mathbb{R}_{\geq 0}$ denotes the set of all real numbers that are ≥ 0. That is, $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$.

Then,

$$\mathbb{R}_{\geq 0} \cap \mathbb{Q} = \{x \in \mathbb{Q} \mid x \geq 0\};$$

this is the set of all rational numbers that are ≥ 0.

Also,

$$\mathbb{Q} \setminus \mathbb{R}_{\geq 0} = \{x \in \mathbb{Q} \mid x < 0\};$$

this is the set of all negative rational numbers.

Let {primes} mean the set of all primes (= prime numbers). Let {even integers} mean the set of all even integers (this is also called $2\mathbb{Z}$). Then,

$$\{\text{even integers}\} \cap \{\text{primes}\} = \{\text{even primes}\} = \{2\},$$

since we have seen that the only even prime is 2. On the other hand,

$$\{\text{primes}\} \setminus \{\text{even integers}\} = \{\text{odd primes}\} = \{3,5,7,11,13,17,\ldots\}.$$

Also,

$$\begin{aligned}\{\text{even integers}\} \triangle \{\text{primes}\} &= \{\text{numbers that are even or prime but not both}\} \\ &= \{\text{odd primes}\} \cup \{\text{even non-primes}\};\end{aligned}$$

this is a weird set that could be written as $\{\ldots,-6,-4,-2,0,3,4,5,6,7,8,10,11,12,13,14,16,\ldots\}$, but good luck figuring out the definition of the set from this expression!

Here are some rules for sets and their different combinations:

Theorem 0.1.8 (rules for set operations)**.** Let S, T and V be three sets. then:

(a) We have $S \cap T \subseteq S \subseteq S \cup T$ and $S \cap T \subseteq T \subseteq S \cup T$.

(b) We have $S \setminus T \subseteq S$ and $(S \setminus T) \cap T = \varnothing$.

(c) We have $S \setminus (S \setminus T) = S \cap T$.

(d) We have $(S \cup T) \cup V = S \cup (T \cup V)$.

(e) We have $(S \cap T) \cap V = S \cap (T \cap V)$.

(f) We have $S \cup (T \cap V) = (S \cup T) \cap (S \cup V)$.

(g) We have $S \cap (T \cup V) = (S \cap T) \cup (S \cap V)$.

(h) We have $(S \bigtriangleup T) \bigtriangleup V = S \bigtriangleup (T \bigtriangleup V)$.

(i) We have $S = (S \cap T) \cup (S \setminus T)$.

(j) We have $S \cup T = S$ if and only if $T \subseteq S$.

(k) We have $S \cap T = S$ if and only if $S \subseteq T$.

Proof. All of these can be proved mechanically by recalling that two sets are equal if they have the same elements. For example, let us do part **(i)**:

(i) We must prove that $S = (S \cap T) \cup (S \setminus T)$. In other words, we must prove that the sets S and $(S \cap T) \cup (S \setminus T)$ have the same elements. In other words, we must prove that any object x belongs to S if and only if it belongs to $(S \cap T) \cup (S \setminus T)$. So let x be some object. This object x can either belong or not belong to S, and it can either belong or not belong to T. So we have four options, and in each of these four cases we can mechanically check whether x belongs to $(S \cap T) \cup (S \setminus T)$ by recalling the definition:

$x \in S$?	$x \in T$?	$x \in S \cap T$?	$x \in S \setminus T$?	$x \in (S \cap T) \cup (S \setminus T)$?
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	F	F

This truth table (again, "T" and "F" stand for "true" and "false") shows that the cases in which x belongs to $(S \cap T) \cup (S \setminus T)$ are exactly the same cases as the ones in which x belongs to S. So S and $(S \cap T) \cup (S \setminus T)$ have the same elements. In other words, $S = (S \cap T) \cup (S \setminus T)$. This proves part **(i)**.

Alternatively, part **(i)** can be proved in a logical way. Indeed, the definition of union yields

$$
\begin{aligned}
&(S \cap T) \cup (S \setminus T) \\
&= \left\{ x \mid \underbrace{x \in S \cap T}_{\iff (x \in S \text{ AND } x \in T)} \quad \text{OR} \quad \underbrace{x \in S \setminus T}_{\iff (x \in S \text{ AND } (\text{NOT } x \in T))} \right\} \\
&= \left\{ x \mid \underbrace{(x \in S \text{ AND } x \in T) \text{ OR } (x \in S \text{ AND } (\text{NOT } x \in T))}_{\iff (x \in S \text{ AND } (x \in T \text{ OR } (\text{NOT } x \in T)))} \right\} \\
&= \left\{ x \mid x \in S \text{ AND } \underbrace{(x \in T \text{ OR } (\text{NOT } x \in T))}_{\text{this statement is always true}} \right\} \\
&= \{ x \mid x \in S \} = S.
\end{aligned}
$$

Some other parts of the theorem can also be proved in such ways. □