Math 220 Fall 2021, Lecture 6: Sets

0.1. Logical connectives (cont'd)

0.1.1. Equivalence ($\Longleftrightarrow$) (cont'd)

There are some more relations between our connectives:

Theorem 0.1.1 (rules for connectives)**.** Let P, Q and R be three propositions. Then:

(a) We have $P \iff \text{NOT } (\text{NOT } P)$.
(b) We have $(P \text{ OR } Q) \iff (Q \text{ OR } P)$.
(c) We have $(P \text{ AND } Q) \iff (Q \text{ AND } P)$.
(d) We have $(P \text{ AND } (Q \text{ AND } R)) \iff ((P \text{ AND } Q) \text{ AND } R)$.
(e) We have $(P \text{ OR } (Q \text{ OR } R)) \iff ((P \text{ OR } Q) \text{ OR } R)$.
(f) We have $(P \Longrightarrow Q) \iff ((\text{NOT } P) \text{ OR } Q)$.
(g) We have $Q \implies (P \Longrightarrow Q)$.
(h) We have $(P \text{ AND } Q) \implies P$.
(i) We have $P \implies (P \text{ OR } Q)$.
(j) We have $(P \implies Q) \iff ((\text{NOT } Q) \implies (\text{NOT } P))$.

Note that in general, parentheses are important: "$1 = 0 \text{ AND } 2 = 2 \text{ OR } 3 = 3$" can be either true or false depending on how to read it:

$$(1 = 0 \text{ AND } 2 = 2) \text{ OR } 3 = 3 \qquad \text{is true,} \qquad \text{but}$$
$$1 = 0 \text{ AND } (2 = 2 \text{ OR } 3 = 3) \qquad \text{is false.}$$

So don't write things like "$1 = 0 \text{ AND } 2 = 2 \text{ OR } 3 = 3$" !

However, part **(d)** of the above theorem shows that you can drop the parentheses when the only connective around is AND. So you can write "$P \text{ AND } Q \text{ AND } R$" without worrying about ambiguity.

Proof of the Theorem. Each part of the theorem can be checked mechanically by comparing the truth tables. Let us just do it for part **(d)**:

P	Q	R	$(P \text{ AND } Q) \text{ AND } R$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

and

P	Q	R	$P \text{ AND } (Q \text{ AND } R)$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

.

The rest is equally easy and left to the reader. $\square$

Note that the connectives AND, OR, XOR and $\Longleftrightarrow$ are symmetric (e.g., $\Longleftrightarrow$ being symmetric means that "$P \Longleftrightarrow Q$" means the same as "$Q \Longleftrightarrow P$"). However, $\Longrightarrow$ is not symmetric ("$P \Longrightarrow Q$" is not the same as "$Q \Longrightarrow P$"), which is why we have introduced the connective $\Longleftarrow$.

0.2. Sets

The notion of a "set" is one of the most fundamental concepts of mathematics. It is not formally defined, but you should understand a **set** to be a well-defined collection of objects. The objects contained in a set will be called the **elements** (or **members**) of the set. We write "$x \in S$" to say "x is contained in S", and we write "$y \notin S$" to say "y is not contained in S".

Here are some **examples of sets**:

- The set $\{1, 2, 3\}$. This set contains only 3 elements: 1, 2 and 3. So, for example, $2 \in \{1, 2, 3\}$ but $5 \notin \{1, 2, 3\}$.
- The set $\mathbb{N}$. This is defined to be the set of all nonnegative integers. Its elements are $0, 1, 2, 3, \ldots$; I cannot list them all because there are infinitely many of them. This set can also be written $\{0, 1, 2, 3, \ldots\}$. For example, $2 \in \mathbb{N}$ but $-5 \notin \mathbb{N}$.
- The set $\mathbb{Z}$. This is defined to be the set of all integers. It can also be written $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ or $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- The set $\mathbb{Q}$. This is defined to be the set of all rational numbers (i.e., of all fractions $\frac{a}{b}$, where a and b are integers and $b \neq 0$). How would you list its elements? Is it even possible to list its elements? Maybe

  $$\left\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \ldots, \frac{-3}{2}, \frac{-2}{2}, \frac{-1}{2}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots, \ldots, \frac{-3}{3}, \frac{-2}{3}, \ldots\ldots\ldots\right\}.$$

  But this is more confusing than helpful. Note that $\frac{3}{2} \in \mathbb{Q}$ but $\sqrt{2} \notin \mathbb{Q}$ and $\pi \notin \mathbb{Q}$.
- The set $\mathbb{R}$. This is the set of all real numbers, including rational numbers but also stranger things like $\sqrt{2}$ and π and $\sqrt{2}^{\pi}$ and even wilder creatures.
- The one-element set $\{1\}$. It contains 1 and nothing else.
- The empty set $\{\}$. It contains nothing (i.e., there is no object that is contained in this set). This set is also denoted $\varnothing$.

- The set $\{\{1,2\}, \{1,3\}, \{2,3\}\}$. This is a set whose elements are themselves sets – namely, the three sets $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$. These elements themselves contain elements. In particular, the numbers 1, 2 and 3 are not elements of the set $\{\{1,2\}, \{1,3\}, \{2,3\}\}$, but rather are elements of its elements.
- You can mix objects of different kinds in a single set. For example, $\{1, \pi, \{1,2\}, \mathbb{N}\}$ is a perfectly valid 4-element set.

As you have guessed from these examples, there are two basic ways to describe a set:

Way 1 is to list all elements. If $a_1, a_2, \ldots, a_k$ are some objects, then "$\{a_1, a_2, \ldots, a_k\}$" means the set that contains these exact objects $a_1, a_2, \ldots, a_k$ and nothing else. This is how $\{1,2,3\}$ and $\{1\}$ and $\{\}$ are understood. The notation $\{0,1,2,3,\ldots\}$ is also an instance of this pattern, at least if we extend this pattern to infinite lists of objects.

Keep in mind that the "..." notation is informal and somewhat dangerous. Handle it with care. When you write $\{1,2,\ldots,49\}$, it is fairly clear that you mean the set of the first 49 positive integers. When you write $\{1,4,\ldots,49\}$, do you mean the set $\{1,4,7,10,13,16,19,22,25,28,31,34,37,40,43,46,49\}$ or do you mean the set $\{1,4,9,16,25,36,49\}$? If such ambiguity is possible, you should be clearer about what set you want. For instance, if you mean the former set, you can write

$$\{3 \cdot 0 + 1, \ \ 3 \cdot 1 + 1, \ \ 3 \cdot 2 + 1, \ \ \ldots, \ \ 3 \cdot 16 + 1\}.$$

If you mean the latter set, you can write $\left\{1^2, 2^2, 3^2, \ldots, 7^2\right\}$.

A variant of Way 1 is writing something like "$\left\{x^2 \mid x \in \mathbb{N}\right\}$". This should be read as "the set that contains all the squares x^2 with $x \in \mathbb{N}$". So it is the set consisting of 0^2, 1^2, 2^2, 3^2, ... In general, the notation

$$\text{" } \{(\text{some expression that involves } x) \mid x \in S\} \text{ "}$$

(where S is some already existing set) means the set of all values that this expression can take when x runs through S. So you substitute for x each possible element of S (one at a time), and you collect all possible values that the expression can take. The resulting set is this collection.

For example,

$$\left\{x^2 \mid x \in \{0,1,2\}\right\} = \left\{0^2, \ 1^2, \ 2^2\right\} = \{0,1,4\}$$

and

$$\{-x \mid x \in \{0,1,2\}\} = \{-0, \ -1, \ -2\} = \{0, \ -1, \ -2\}$$

and

$$\left\{x^2 + x \mid x \in \{0,1,2\}\right\} = \left\{0^2 + 0, \ 1^2 + 1, \ 2^2 + 2\right\} = \{0,2,6\}.$$

This notation can be extended to expressions that involve multiple variables (and you don't have to call them x). For instance,

$$\left\{y^2+z^2 \mid y \in \{0,1,2\} \text{ and } z \in \{0,1,2\}\right\}.$$

This is the set consisting of all possible values of y^2+z^2 when y is an element of $\{0,1,2\}$ and z is an element of $\{0,1,2\}$. To list its elements, we make a table of these values:

$y \backslash z$	0	1	2
0	0	1	4
1	1	2	5
2	4	5	8

Thus, our set is $\{0,1,4,1,2,5,4,5,8\}$. Note that y and z are allowed to be equal.

Way 2 to describe a set is by filtering elements from an existing set through some condition. For example,

$$\{x \in \mathbb{R} \mid 1 < x < 2\}$$

is the set of all $x \in \mathbb{R}$ (that is, of all real numbers x) that satisfy $1 < x < 2$. In other words, it is the set of all real numbers that lie between 1 and 2 exclusively. For example, 1.5 and 1.7 and $\sqrt{2}$ belong to this set, but 2 and 1 and π and 75 and -13 do not. The general version of this notation is

$$\text{“}\{x \in S \mid \text{some condition}\}\text{”},$$

where S is a given set and where "some condition" is some well-defined predicate that involves the variable x. It means the set of all $x \in S$ that satisfy this condition.

Some **examples** of this:

- $\{x \in \mathbb{Z} \mid 1 < x < 7\} = \{2,3,4,5,6\}$.
- $\{x \in \mathbb{Z} \mid 1 \leq x \leq 7\} = \{1,2,3,4,5,6,7\}$.
- $\{x \in \mathbb{Q} \mid 1 < x < 7\}$ is a set that contains $2,3,4,5,6$ but also infinitely many other numbers (e.g., it contains $\dfrac{3}{2}$).
- $\{x \in \mathbb{Q} \mid 7 < x < 1\} = \{\} = \varnothing$.
- $\{y \in \mathbb{Z} \mid y \text{ is divisible by 3 and divides } 12\} = \{3,6,12,-3,-6,-12\}$.
- $\{y \in \mathbb{Q} \mid y \text{ is divisible by 3 and divides } 12\}$ is meaningless, because "y is divisible by 3" only has a well-defined meaning when y is an integer.
- $\{x \in \{y \in \mathbb{Z} \mid 1 < y < 7\} \mid x \text{ is even}\} = \{2,4,6\}$. Of course, this is better written as $\{x \in \mathbb{Z} \mid 1 < x < 7 \text{ and } x \text{ is even}\}$, but $\{x \in \{y \in \mathbb{Z} \mid 1 < y < 7\} \mid x \text{ is even}\}$ is a perfectly valid notation. Just be careful to not write $\{x \in \{x \in \mathbb{Z} \mid 1 < x < 7\} \mid x \text{ is even}\}$. Variables should not be reused in the same formula.

- $\{x \mid x \text{ is a prime number}\}$ is, strictly speaking, not an instance of our notation, because there is not set on the left of the $|$. But it's clear and unambiguous: it just means the set of all prime numbers. So we allow it as well. However, $\{x \mid x^2 = x\}$ is not unambiguous, because I'm not saying what x is and it does not follow from the context. Maybe I mean real numbers, but maybe I mean matrices.

- In geometry, sets defined by filtering are called "loci" (this is the plural of "locus"). For example, "The locus of all points that are at the same distance from a given point P and a given line ℓ is a parabola". In our language, this is simply saying that

  $$\text{the set } \{Q \in \text{Pts} \mid \text{dist}(Q, P) = \text{dist}(Q, \ell)\} \text{ is a parabola,}$$

  where Pts is the set of all points in the plane.