Math 220 Fall 2021, Lecture 5: Logical connectives, sets and quantifiers

0.1. Logical connectives (cont'd)

Last time, we learned about AND, OR and XOR. These operations are called **logical connectives**, because they connect propositions to form new propositions. Let us see some more logical connectives.

0.1.1. Implication (IMPLIES, aka $\Longrightarrow$) (cont'd)

We introduced the "$\Longrightarrow$" connective (also known as "IMPLIES") last time. Let us recall what it says: If P and Q are two propositions, then "$P \Longrightarrow Q$" (pronounced "if P, then Q") is a proposition that is

- true when P is false,
- true when Q is true,
- false otherwise.

In fact, you can think of "$P \Longrightarrow Q$" as a promise of Q contingent on the condition P. Thus, if P is false, then the promise does not come due, because it is contingent on something that is not satisfied. If Q is true, then the promise is satisfied. Only in the case when P is true and Q is false is the promise actually broken.

Once again, here is the truth table for $P \Longrightarrow Q$:

P	Q	$P \Longrightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Note that this is precisely the truth table of "$(\text{NOT } P)$ OR Q". This is not surprising because, as we said, the two ways how a promise of Q contingent on P can be fulfilled are

- either by ensuring that the condition P is false,
- or by paying the promise (i.e., ensuring that Q is true).

Note that the "OR" in $(\text{NOT } P)$ OR Q is not exclusive.

A statement of the form "$P \Longrightarrow Q$" is called an **implication** (or sometimes a **conditional**). We refer to P as the **hypothesis** or the **condition**, and to Q as the **conclusion**. If P is false, then we say that "$P \Longrightarrow Q$" is **vacuously true**.

The statement "$P \implies Q$" is often pronounced/written "if P, then Q" or "if P, Q" (used mostly when P and Q consist of words, not of symbols) or "P implies Q" or "Q if P" or "P is sufficient for Q" or "Q is necessary for P" or "P only if Q".

Some **examples** of implications:

- "Let n and m be two integers. If $n < m$, then $-n > -m$."

  The second sentence here is an implication ("If ..., then ..."). It is always true.

- "Let n and m be two integers. If $n \leq m$ and $n \geq m$, then $n = m$."

  Again, this is true.

- "Let n and m be two integers. If $n \leq m$ and $-n \geq -m$, then $n = m$."

  This is again an implication, but this one is false (e.g., for $n = 0$ and $m = 1$).

- "Let n and m be two integers. If $n < m$ and $n > m$, then $0 = 1$."

  This is again an implication. This one is true, and indeed vacuously true, because the condition ("$n < m$ and $n > m$") is always false. It doesn't matter that $0 = 1$ is false.

- "Let n and m be two integers. If $n < m$ and $n > m$, then $0 = 0$."

  This is still vacuously true, but it's also true because $0 = 0$ is true.

- "Let n and m be two integers. If n and m are odd, then nm is odd."

  This is true, as we have proved a few lectures ago.

- "Let n and m be two integers. If $n = m$, then $n^2 = m^2$."

  This is true, and indeed is a consequence of the **principle of substitution**, which says that equal things can be substituted for one another.

  (Be careful: Substitution requires parentheses. For example, $2 = 1 + 1$, but this doesn't mean that we can literally replace the "2" in "$2 \cdot 5$" by a "$1 + 1$" and obtain "$1 + 1 \cdot 5$". Instead you have to put parentheses around the result, so you obtain "$(1 + 1) \cdot 5$". But in our case, n and m are just single letters, so we don't need parentheses.)

- "Let n and m be two integers. If $n^2 = m^2$, then $n = m$."

  This is false, e.g., for $n = 5$ and $m = -5$, because here $n^2 = m^2$ is true but $n = m$ is not.

- "Let n and m be two integers. If $n = m$, then $n^3 = m^3$."

  This is true, again because of the principle of substitution.

- "Let n and m be two integers. If $n^3 = m^3$, then $n = m$."

  This is true. (Indeed, it is easy to verify that $k^3 < (k+1)^3$ for any integer k (because $(k+1)^3 - k^3 = 3k^2 + 3k + 1 = 3\left(k + \frac{1}{2}\right)^2 + \frac{1}{4} > 0$); in other words,

  $$\cdots < (-2)^3 < (-1)^3 < 0^3 < 1^3 < 2^3 < \cdots ;$$

  but this implies that all the cubes $\ldots, (-2)^3, (-1)^3, 0^3, 1^3, 2^3, \ldots$ are distinct.)

- "Let n be an integer. If $n > 1$, then there is a prime that divides n."

  This is true, and we proved this a while ago.

There is one further related connective, which is called "$\Longleftarrow$". Can you guess its meaning?

If P and Q are two propositions, then "$P \Longleftarrow Q$" is simply another way to say "$Q \Longrightarrow P$". This is pronounced "P is implied by Q".

0.1.2. Equivalence ($\Longleftrightarrow$)

Finally, here is one of the most important connectives around: equivalence.

Let P and Q be two propositions. Then, "$P \Longleftrightarrow Q$" (pronounced "P if and only if Q" or "P is equivalent to Q" or "P and Q are equivalent") is a new proposition, which is

- true when P and Q are both true;
- true when P and Q are both false;
- false in the other cases.

In other words, its truth table is

P	Q	$P \Longleftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

So "$P \Longleftrightarrow Q$" means that P and Q have the same truthfulness. Thus, if P and Q are equivalent, then we can replace P by Q or vice versa. Thus the name "equivalent" (Latin for "equal in value").

A statement of the form "$P \Longleftrightarrow Q$" is called an **equivalence**.

Some **examples** of equivalence:

- "Let n and m be two integers. Then, we have $n = m$ if and only if $n^2 = m^2$."

  This (specifically, the second sentence) is an equivalence. It is false for $n = 5$ and $m = -5$, since $n = m$ is false but $n^2 = m^2$ is true.

- "Let n and m be two integers. Then, we have $n = m$ if and only if $n^3 = m^3$."

  This is again an equivalence. It is true.

- "We have $1 + 2 = 3$ if and only if $1 + 1 = 2$."

  This is true, because both $1 + 2 = 3$ and $1 + 1 = 2$ are true.

- "We have $1 + 2 = 3$ if and only if $1 + 1 = 3$."

  This is false, because $1 + 2 = 3$ is true but $1 + 1 = 3$ is false.

- "We have $1 + 2 = 5$ if and only if $1 + 1 = 3$."

  This is true, because both $1 + 2 = 5$ and $1 + 1 = 3$ are false.

Since equivalent propositions have the same truthfulness, we will treat them as being the same. This allows us to get some order into the jungle of logical connectives, because two connectives with the same truth table are equivalent and thus can be equated. There are $2^4 = 16$ ways to fill in a truth table; thus, there are 16 logical connectives altogether.

We know that AND, OR, XOR, $\Longrightarrow$, $\Longleftarrow$ and $\Longleftrightarrow$ are 6 of these 16 connectives. Another is NAND, which is defined by

$$(P \text{ NAND } Q) \iff (\text{NOT } (P \text{ AND } Q)).$$

Its truth table is

P	Q	P NAND Q
T	T	F
T	F	T
F	T	T
F	F	T

.

It is usually pronounced as "P and Q are not both true".

Another connective is NOR, which is defined by

$$(P \text{ NOR } Q) \iff (\text{NOT } (P \text{ OR } Q)).$$

Its truth table is

P	Q	P NOR Q
T	T	F
T	F	F
F	T	F
F	F	T

.

It usually pronounced as "neither P nor Q is true".

We can build new connectives either by specifying a truth table, or by composing old connectives. For example, we can define a new connective ANDN by

$$(P \text{ ANDN } Q) \iff ((\text{NOT } P) \text{ AND } (\text{NOT } Q)).$$

What is the truth table of ANDN ?

P	Q	P ANDN Q
T	T	F
T	F	F
F	T	F
F	F	T

.

This is the same truth table as the one of NOR ! Thus, ANDN and NOR are the same connective. In other words, for any two propositions P and Q, we have the equivalence

$$(P \text{ ANDN } Q) \iff (P \text{ NOR } Q).$$

Let us state this without the use of ANDN and NOR:

$$((\text{NOT } P) \text{ AND } (\text{NOT } Q)) \iff (\text{NOT } (P \text{ OR } Q)).$$

Similarly, it can be shown that

$$((\text{NOT } P) \text{ OR } (\text{NOT } Q)) \iff (\text{NOT } (P \text{ AND } Q)).$$

Let us state these two equivalences as a theorem:

Theorem 0.1.1 (de Morgan's laws for AND and OR)**.** Let P and Q be two propositions. Then, we have the equivalences

$$((\text{NOT } P) \text{ AND } (\text{NOT } Q)) \iff (\text{NOT } (P \text{ OR } Q))$$

and

$$((\text{NOT } P) \text{ OR } (\text{NOT } Q)) \iff (\text{NOT } (P \text{ AND } Q)).$$

Proof. For both equivalences, simply compare the truth tables. The best way to do this is using multi-column truth tables:

P	Q	$(\text{NOT } P) \text{ AND } (\text{NOT } Q)$	$\text{NOT } (P \text{ OR } Q)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

;

P	Q	$(\text{NOT } P) \text{ OR } (\text{NOT } Q)$	$\text{NOT } (P \text{ AND } Q)$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

.

$\square$

There are some more relations between our connectives:

Theorem 0.1.2 (rules for connectives)**.** Let P, Q and R be three propositions. Then:

(a) We have $P \iff \text{NOT } (\text{NOT } P)$.
(b) We have $(P \text{ OR } Q) \iff (Q \text{ OR } P)$.
(c) We have $(P \text{ AND } Q) \iff (Q \text{ AND } P)$.
(d) We have

$$(P \text{ AND } (Q \text{ AND } R)) \iff ((P \text{ AND } Q) \text{ AND } R).$$

(e) We have

$$(P \text{ OR } (Q \text{ OR } R)) \iff ((P \text{ OR } Q) \text{ OR } R).$$

[To be continued next time.]