Math 220 Fall 2021, Lecture 4: Logical connectives

Last time, we proved the proposition that there are infinitely many primes.

This was a neat argument that looks simple in hindsight. On a closer look, it uses a bunch of facts that themselves need to be proved:

- that there **exists** a smallest positive divisor d of N that is greater than 1.
- the uniqueness of remainders.
- the transitivity of the $<$ relation: If $a < b$ and $b < c$, then $a < c$. (We used this implicitly when we said "even-smaller".)

This is unsurprising: We cannot get anything interesting from nothing. Any proof relies on some existing knowledge that is taken for granted. Ideally, of course, the facts that are used in the proof of a proposition should be more basic, more elementary, more certain than the proposition itself; in particular, you cannot use the proposition itself in its own proof. (This is called "circular reasoning" and is just as wrong in real life as in maths.)

When doing formal proofs, you have to decide which propositions you take for granted. These propositions are called "**axioms**". They are basically standing assumptions on the world that you are talking about. For instance, one such proposition is the transitivity of the $<$ relation (i.e., "If a, b, c are numbers with $a < b$ and $b < c$, then $a < c$"). Another such proposition is the commutativity of addition (i.e., "If a, b are numbers, then $a + b = b + a$"). Actually, both of these propositions can be proved using more basic axioms, but we will not do so. Instead, we will try to prove some more interesting facts.

0.1. Logical connectives

Let us again take a look at our above proof of the fact that there are infinitely many primes. What is this proof made of? Of course, it is made of sentences, but what kind of sentences?

- Some of the sentences look like propositions or predicates. For example, "This prime p equals none of $p_1, p_2, \ldots, p_k$" sounds like a predicate, because it involves the variables $p, p_1, p_2, \ldots, p_k$. However, all these variables have been introduced before, so they have precise meanings at the time this sentence is said. Thus, this sentence is a proposition in its context. If it were to appear outside of the proof, it would be a predicate.
- Other statements are definitions ("Set $N = p_1 p_2 \cdots p_k + 1$") or specifications ("Let $p_1, p_2, \ldots, p_k$ be finitely many primes").

- Finally, many of the sentences are deduction steps: "This shows that N leaves the remainder 1 when divided by p". The "N leaves the remainder 1 when divided by p" part of this sentence is a proposition, but the "This shows that" part is a signal that this proposition follows from the sentence that comes before it.

Definitions and specifications are boring: Introducing names doesn't by itself convince us of anything. So the "active ingredient" in the proofs must be the propositions and the deduction steps connecting them. Obviously, we are allowed to deduce some propositions from others. Let us try to understand what the rules of this game us.

To state these rules, we first introduce some language for what we can do with propositions.

0.1.1. Negation (NOT)

If P is a proposition, then NOT P is a proposition too. This proposition NOT P is simply saying that P is false. In other words, NOT P is true when P is false, and is false when P is true. So the "NOT" simply flips the truthfulness of P.

For example:

- If P is "$2+2=4$", then NOT P is "$2+2$ is not 4". This is commonly written as "$2+2 \neq 4$".

- If P is "every integer is prime", then NOT P is "not every integer is prime". Some might restate this as "every integer is not prime"; however, this is ambiguous, since it can be understood both as "not every integer is prime" and as "every integer is non-prime". (Cf. "all that glitters is not gold".)

Grammar is tricky here: Sometimes there are several places in a sentence where you can put the word "not", and not all of them have the right effect (which is to flip the truthfulness of P). If you are not sure how to correctly formulate NOT P, you can always formulate it as "It is not true that P". So, for example, if P is "every integer is prime", NOT P is "it is not true that every integer is prime". This is pedantic but isn't wrong.

So let us summarize what the word "NOT" does in a table:

P	NOT P
T	F
F	T

This is called a "**truth table**". It tells us whether NOT P is true or false depending on whether P is true or false. In the left column, you have the possible cases for the truthfulness of P (namely, "T" means that P is true, and "F" means that P is false).

In the right column, you have the corresponding information for NOT P. So the table simply tells you that NOT P is false if P is true, and vice versa.

We call NOT P the **negation** of P. Mathematicians sometimes use the alternative notations $\neg P$ and $\tilde{}P$ for NOT P.

Note that NOT (NOT P) is equivalent to P, meaning that NOT (NOT P) is true when P is true and false when P is false. This should be fairly intuitive given the above definition of NOT P: Flipping the truthfulness twice should bring it back to its original value.

0.1.2. Conjunction (AND)

So when we have one proposition, we can negate it. What can we do with two propositions?

We can combine them in several ways. One is by using the word "AND".

For example, if P is "$2 + 2 = 4$" and Q is "2 is prime", then P AND Q is saying that "$2 + 2 = 4$ and 2 is prime".

So, if P and Q are two propositions, then P AND Q is a new proposition, which says that **both** P and Q are true. Thus, the truth table for AND is the following:

P	Q	P AND Q
T	T	T
T	F	F
F	T	F
F	F	F

The first two columns in this table list all possibilities for P and Q separately to be true or false. The third column tells us whether P AND Q is true or false in these cases.

The proposition P AND Q is called the **conjunction** of P and Q. It is usually denoted $P \wedge Q$ in mathematics. We will try to use the word " AND " instead.

Some examples:

- If P is "3 is positive" and Q is "3 is odd", then P AND Q is "3 is positive and 3 is odd". A shorter way to say this is "3 is positive and odd".
- If P is "Some integers are divisible by 4" and Q is "Some integers are prime", then P AND Q is "Some integers are divisible by 4, and some integers are prime". You **cannot** contract this to "Some integers are divisible by 4 and prime". (Indeed, P AND Q is true, whereas "Some integers are divisible by 4 and prime" is false.) The problem here is that "Some integers" makes our claim existential, and the "some integers" that make P true (such as 8) have nothing to do with the "some integers" that make Q true (such as 3). So, while P AND Q is true, there is no reason to expect that P and Q should be true by virtue of the same integers.

0.1.3. Disjunction (OR)

What else can we do with two propositions? We can combine them using the word "OR".

For example, if P is "3 is positive" and Q is "3 is odd", then P OR Q is "3 is positive or 3 is odd". A shorter way to say this is "3 is positive or odd".

So, if P and Q are two propositions, then P OR Q is a new proposition, which says that **at least one of** P and Q is true. Thus, the truth table for OR is the following:

P	Q	P OR Q
T	T	T
T	F	T
F	T	T
F	F	F

Sometimes, "P OR Q" is pronounced/written "either P or Q". The word "either" usually does not by itself carry any meaning; it just marks the place where the statement P begins. However, sometimes it stands for another kind of "or", namely the "exclusive or" we will see below.

Unfortunately, everyday language has its own, context-dependent understanding of what "or" means, which does not always agree with the logical definition of "or". Instead of meaning "at least one of P and Q holds", it can mean "exactly one of P and Q holds" or even "P and Q cannot both hold". For example, when you hear "you can have either speed or quality", the intended meaning is "you cannot have both speed and quality at the same time". So this is much different from the logical meaning of "or".

The proposition P OR Q is called the **disjunction** of P and Q. It is usually denoted $P \vee Q$ in mathematics. We will try to use the word " OR " instead.

As mentioned above, there is also an "exclusive or", which is written P XOR Q, which is saying that **exactly one** of P and Q is true. So the truth table for XOR is:

P	Q	P XOR Q
T	T	F
T	F	T
F	T	T
F	F	F

As I said, mathematicians never just say "or" for "XOR".

0.1.4. Implication (IMPLIES, aka $\Longrightarrow$)

Here is one more thing we can do with two propositions P and Q: We can say "If P is true, then so is Q" (or, short: "If P, then Q"). For example, if P is "2 is even" and Q is "2 is prime", then we can say "If 2 is even, then 2 is prime". Of course, this is a stupid statement, but it is true.

In mathematical language, "if P, then Q" is commonly written as "$P \Longrightarrow Q$" (or P IMPLIES Q). The truth table for this operation is:

P	Q	$P \Longrightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Thus, $P \Longrightarrow Q$ is

- true if P is false,
- true if Q is true,
- false in the remaining case.