Math 220 Fall 2021, Lecture 2: Propositions

Recall **Conjecture 1** from last time: "Let n be a positive integer. Then, n has at least as many 1-odd posdivs (= positive divisors) as it has 3-odd posdivs."

By experiment, we checked that this is true for as many n as we could run on our computer. Based on our experience with Conjecture 2, this is not really sufficient.

Let me try to convince you differently that Conjecture 1 is true. Instead of running experiments, I will try to argue its truth. To wit, I will pair up each 3-odd posdiv of n with a 1-odd posdiv of n. Some 1-odd posdivs might stay unpaired, but in either case the number of 3-odd posdivs will be less or equal to the number of 1-odd posdivs, because all 3-odd posdivs will be paired.

How do I pair them up? For each 3-odd posdiv d of n, I pick the smallest 3-odd posdiv q of d, and pair up d with $\frac{d}{q}$. It is not hard to check that $\frac{d}{q}$ is 1-odd. Indeed, if $q = 4u + 3$ and $\frac{d}{q} = 4p + r$ with $r \in \{0, 1, 2, 3\}$, then

$$d = q \cdot \frac{d}{q} = (4u+3)(4p+r) = \underbrace{16up + 4ur + 12p}_{\text{divisible by 4}} + 3r,$$

which easily gives $r = 1$ (since the cases $r = 0$, $r = 2$ and $r = 3$ quickly lead to expressions that contradict d being 3-odd); this means that $\frac{d}{q}$ is 1-odd.

So we can assign to each 3-odd posdiv d of n the 1-odd posdiv $\frac{d}{q}$ of n. Looks good, right?

Let us see this on an example:

Example: Let $n = 225 = 3^2 5^2$. The posdivs of n are

1	3	9
5	15	45
25	75	225

Our assignment sends

$$3 \to 1, \qquad 15 \to 5, \qquad 75 \to 25.$$

Another example: Let $n = 21$. The posdivs of n are

1	3
7	21

Our assignment sends

$$3 \to 1, \qquad 7 \to 1.$$

So our pairing is not actually a pairing! We have "paired up" two different numbers with one and the same partner.

What happened? We got fooled by a word, namely the word "pairing". It sounds like assigning a 1-odd posdiv to each 3-odd posdiv would constitute a "pairing", but implicitly the notion of a "pairing" requires more: it requires the partners to be distinct.

So words are dangerous when they have multiple meanings or when they aren't properly defined.

[The same problem underlies a famous joke:
Theorem. A dog has 9 legs.
Proof. No dog has 5 legs.
A dog has 4 more legs than no dog.
$\Longrightarrow$ A dog has 9 legs.]

So, words are slippery. But words are too useful to get discarded completely. We do want to reason with them, just correctly.

We need a standard of convincingness for a mathematical argument. Computation is not enough.

So let us try to set a standard for what constitutes a valid mathematical argument that convinces us that a certain statement is correct. This standard is known as "proof".

0.1. Propositions

Definition 0.1.1. A **proposition** is a clear, objective and unambiguous statement that is either true or false. Sometimes, we will just say "statement" for "proposition".

Some **examples of propositions**:

- "$2 + 3 = 5$" is a proposition. This one happens to be true.
- "$2 + 3 = 6$" is a proposition. This one happens to be false.
- "If n is a positive integer, then there are at least as many 3-odd primes up until n as there are 1-odd primes up until n" is a proposition. This is our Conjecture 2 from last time, and as we have seen, it is false.
- "If n is a positive integer, then n has at least as many 1-odd posdivs as it has 3-odd posdivs" is a proposition. This is our Conjecture 1, and we don't know whether it is true or false.
- "There are no positive integers x, y, z and $n > 2$ such that $x^n + y^n = z^n$" is a proposition. This is a rather famous result, known as Fermat's Last Theorem. Originally claimed by Fermat around 1637 and only proved in 1995 by Wiles and Taylor.

- "Every even integer greater than 2 is the sum of two prime numbers". This is a proposition called Goldbach's conjecture; no one knows whether it is true or false.
- "If a, b, c are the lengths of the sides of a right-angled triangle, then $a^2 + b^2 = c^2$". This is a proposition. It is false because we haven't said which of a, b, c is the hypotenuse. For example, there is a right-angled triangle with sides $5, 3, 4$, but $5^2 + 3^2 \neq 4^2$.
- "If a and b are the lengths of the legs of a right-angled triangle, and c is the length of the hypotenuse of this triangle, then $a^2 + b^2 = c^2$". This is a true proposition, known as Pythagoras's theorem.
- "The altitudes of a triangle have a point in common". This is another true proposition, and appears in Euclid's Elements.
- "If $f : [a, b] \to \mathbb{R}$ is a differentiable function on an interval $[a, b]$, then $\int_a^b f'(x)\,dx = f(b) - f(a)$" is a true proposition from analysis (called the Fundamental Theorem of Calculus).
- "A polynomial of degree n with real coefficients has no more than n roots" is a true proposition.
- "Consider a rectangular table (= matrix) with real entries such that in each row, the entries increase from left to right (to be specific, weakly increase – i.e., adjacent entries can be equal). You sort all numbers in each column so that in each column, the numbers now increase from top to bottom. Then, the numbers still increase from left to right in each row".

  Example:

  $$\begin{matrix} 2 & 3 & 4 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{matrix} \to \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{matrix}$$

  This is a proposition. Is it true (not just in our example)? Yes, and it is known as the "non-messing-up lemma".

Examples of non-propositions:

- "$a^2 + b^2 = c^2$" is not a proposition, since it is not explained what a, b and c are.
- "If n is a positive integer, potatoes" is not a proposition either.
- "Is 31 a prime number?" is not a proposition, since it's a question rather than a statement.

- "Solve $x^3 - x^2 - 2x + 2 = 0$" is not even a well-defined question, let alone a proposition. To make it a well-defined question, we would have to specify where are looking: should x be an integer? a real number? a matrix?

- "It has rained yesterday" is not a proposition either, since its meaning depends on when you are reading it.

- "It has rained on 2021-09-21 in Philadelphia" is a proposition, although not in the purview of mathematics.

- "Algebra is useful" is not a proposition, since neither "algebra" nor "useful" has an agreed-upon definition.

- "This proposition is false". This one is tricky. It looks like a legitimate proposition. But if you think about whether it is true or false, you run into trouble. If it is true, then it mistakenly claims itself to be false, which means that it is false. If it is false, then it correctly claims it to be false, which makes it true. Thus we get a paradoxon: whether we believe it is true or false, we get a problem. So it can neither be true nor false.

  But the real problem with this "proposition" is that it is self-referential (it talks about itself). A proposition has to talk about well-defined things, things that "already" are there before this proposition has been stated. It cannot take its own existence for granted.

  Thus, "this proposition is true" is similarly not a proposition, even though it would cause no such paradoxies.

Definition 0.1.2. A **predicate** is a statement that is like a proposition, except that it involves some variables, and its truth or falsity might depend on the values of these variables. For each assignment of values to the variables, it has to be either true or false.

Examples:

- "n is a perfect square" is not a proposition, but it is a predicate with variable n. It is true for $n = 4$ and false for $n = 5$.

- "$a^2 + b^2 = c^2$" is not a proposition, but it is a predicate with variables a, b, c. It is true, e.g., if a and b are the legs .... and in many other cases.

- A proposition counts as a predicate, too (a predicate with 0 variables).

We will often use function-like notation for predicates. So, for example, we might give the predicat "n is a perfect square" the name $P(n)$. Then, $P(5)$ shall mean the proposition "5 is a perfect square".

0.2. Proofs

We don't just want to talk about propositions; we want to tell whether they are true or false. As we saw, examples sometimes suffice but not always, whereas verbal arguments can be treacherous.

In mathematics, the standard for accepting a proposition as true is "proof".