**Math 220-002 Fall 2021 (Darij Grinberg): homework set 4**
due date: Monday 2021-11-09 at noon on gradescope (
`https://www.gradescope.com/courses/313263` ).
Please solve only **4 of the 6 exercises**.

This homework set is mostly about divisibility and congruence.

**Exercise 1.** Let $a, b \in \mathbb{Z}$. Prove the following:
**(a)** If $a \mid b$, then $a^k \mid b^k$ for each $k \in \mathbb{N}$.
**(b)** Let $n \in \mathbb{Z}$. If $a \equiv b \mod n$, then $a^k \equiv b^k \mod n$ for each $k \in \mathbb{N}$.

**Exercise 2.** Let $n, d, a, b \in \mathbb{Z}$, and assume that $d \neq 0$ and $da \equiv db \mod dn$.
**(a)** Prove that $a \equiv b \mod n$.
**(b)** Show by an example that $a \equiv b \mod dn$ is not necessarily true (i.e., we cannot simply cancel the $d$ from $da$ and $db$ while leaving the $dn$ unchanged).

**Exercise 3.** Let $n$ be any integer. Prove the following:
**(a)** If $n$ is odd, then $8 \mid n^2 - 1$.
**(b)** If $3 \nmid n$, then $3 \mid n^2 - 1$.
[**Hint:** In part **(a)**, write $n$ as $2k + 1$. In part **(b)**, write $n$ as $q \cdot 3 + r$ and consider the possible values for $r$.]

**Exercise 4.** Prove that $\gcd(15n + 4,\ 12n + 5) = 1$ for each $n \in \mathbb{Z}$.

Two primes that differ by 2 are called *twin primes*. (For instance, 17 and 19 are twin primes.) An infamous open problem in number theory asks whether there are infinitely many "twin primes". A much easier variant of this question asks how many "double-twin primes" (i.e., primes $p$ such that both $p - 2$ and $p + 2$ are primes, so that $p$ belongs to two twin-primes pairs) exist. The answer is, there is exactly one:

**Exercise 5.** Let $p$ be a prime such that $p - 2$ and $p + 2$ are also prime. Prove that $p = 5$.
[**Hint:** Consider the remainders upon division by 6.]

The next exercise is known as the "Chicken McNugget problem"[1]

[1]Quoting AoPS:

> Originally, McDonald's sold its nuggets in packs of 9 and 20. Math enthusiasts were curious to find the largest number of nuggets that could not have been bought with these packs, thus creating the Chicken McNugget Theorem (the answer worked out to be 151 nuggets).

**Exercise 6.** Let $a$ and $b$ be two coprime positive integers. Let $m$ be an integer such that $m > ab - a - b$. Prove that there exist **nonnegative** integers $x$ and $y$ such that $m = xa + yb$.

[**Hint:** First, use Bezout's theorem to argue that there exist integers $x'$ and $y'$ such that $x'a + y'b = 1$. Hence, obtain integers $x''$ and $y''$ such that $x''a + y''b = m$. If these $x''$ and $y''$ are nonnegative, then we are done. If not, we can "tweak" them by adding a multiple $sb$ of $b$ to $x''$ and subtracting the corresponding multiple $sa$ of $a$ from $y''$. Note that $x'' + sb$ will become nonnegative if we pick $s$ large enough, while $y'' - sa$ while become nonnegative if we pick $s$ small enough. Is the "goldilocks zone" (i.e., the set of all $s \in \mathbb{R}$ that are large enough for $x'' + sb \geq 0$ yet small enough for $y'' - sa \geq 0$) sufficiently large as to contain an integer?]