Math 220-002 Fall 2021 (Darij Grinberg): homework set 3
due date: Monday 2021-11-09 at noon on gradescope (
`https://www.gradescope.com/courses/313263`).
Please solve only **4 of the 6 exercises**.

This homework set is devoted to some further applications of induction (including strong induction).

The first exercise is about matrix multiplication. For an introduction to matrix multiplication, see any textbook on linear algebra (e.g., [BoyVan18, §10.1] goes over it in detail). However, all we need for this exercise will be 2×2-matrices, so let us recall how matrix multiplication works for them:

- The product AX of two 2×2-matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ is defined to be $\begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix}$.
- The n-th power A^n of a 2×2-matrix A is defined to be the product $\underbrace{AA \cdots A}_{n \text{ factors}}$.

Exercise 1. **(a)** Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for each positive integer n.

(b) Find a formula for $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^n$ (with a, b, c being real numbers and n being a positive integer).

The next two exercises rely on binomial coefficients. Recall that if n is any real number and $k \in \mathbb{N}$, then the binomial coefficient $\dbinom{n}{k}$ is defined by

$$\binom{n}{k} := \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

When $k = 0$, this equals 1, because both numerator and denominator are empty products. Do **not** use the formula $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$ unless you are sure that its conditions ($n \in \mathbb{N}$ and $0 \leq k \leq n$) are satisfied!

Exercise 2. Prove that every $n \in \mathbb{N}$ and $a \in \mathbb{R}$ satisfy

$$\sum_{k=0}^{n} \binom{k+a}{k} = \binom{n+a+1}{n}.$$

(For example, for $n = 4$ and $a = 2$, this is saying that

$$\binom{2}{0} + \binom{3}{1} + \binom{4}{2} + \binom{5}{3} + \binom{6}{4} = \binom{7}{4}.$$

Keep in mind that a doesn't have to be an integer in general!)

Exercise 3. Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 0, & \text{if } n \neq 0; \\ 1, & \text{if } n = 0. \end{cases}$$

(For example, for $n = 4$, this is saying that $\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 0$.)

Exercise 4. Find the error(s) in the following fake proof:

We claim that $3^n = 1$ for each $n \in \mathbb{N}$.

Proof: We proceed by strong induction on n. So we let $n \in \mathbb{N}$ be arbitrary, and we assume (as the induction hypothesis) that $3^k = 1$ for each $k < n$. We must now prove that $3^n = 1$.

By our induction hypothesis, we have $3^{n-1} = 1$ (since $n - 1 < n$) and $3^{n-2} = 1$ (since $n - 2 < n$). Now, $3^n = \dfrac{\left(3^{n-1}\right)^2}{3^{n-2}}$ (since the laws of exponents yield $\dfrac{\left(3^{n-1}\right)^2}{3^{n-2}} = 3^{2\cdot(n-1)-(n-2)} = 3^n$). In view of $3^{n-1} = 1$ and $3^{n-2} = 1$, this rewrites as $3^n = \dfrac{1^2}{1} = 1$. This completes the induction step, and thus the claim is proved.

Exercise 5. Define a sequence $(t_0, t_1, t_2, \ldots)$ of positive rational numbers recursively by setting

$$t_0 = 1, \qquad t_1 = 1, \qquad t_2 = 1, \qquad \text{and}$$
$$t_n = \frac{1 + t_{n-1}t_{n-2}}{t_{n-3}} \qquad \text{for each } n \geq 3.$$

(So its next entries after t_2 are

$$t_3 = \frac{1 + t_2 t_1}{t_0} = \frac{1 + 1 \cdot 1}{1} = 2;$$
$$t_4 = \frac{1 + t_3 t_2}{t_1} = \frac{1 + 2 \cdot 1}{1} = 3;$$
$$t_5 = \frac{1 + t_4 t_3}{t_2} = \frac{1 + 3 \cdot 2}{1} = 7;$$
$$t_6 = \frac{1 + t_5 t_4}{t_3} = \frac{1 + 7 \cdot 3}{2} = 11,$$

and so on.)

(a) Prove that $t_{n+2} = 4t_n - t_{n-2}$ for each $n \geq 2$.

(b) Prove that t_n is a positive integer for each integer $n \geq 0$.

[**Hint:** Use regular induction for part **(a)** and strong induction for part **(b)**. Note that the "positive" part is clear from the definition, so you only need to prove the "integer" part in **(b)**.]

Exercise 6. Let $m \in \mathbb{N}$. Prove that there exists a way to arrange the first m positive integers $(1, 2, \ldots, m)$ in a row in such a way that the average of two numbers never stands between these two numbers.

(For example, for $m = 8$, one such arrangement is $1, 5, 3, 7, 2, 6, 4, 8$. The arrangement $1, 3, 2, 7, 8, 5, 6, 4$ is invalid because the average of 1 and 5 is 3, which stands between 1 and 5.)

[**Hint:** First show that there is such an arrangement when m is a power of 2 (that is, when $m = 2^n$ for some $n \in \mathbb{N}$). Then, choose a sufficiently large power of 2 and remove all entries larger than m.]

References

[BoyVan18] Stephen Boyd, Lieven Vandenberghe, *Introduction to Applied Linear Algebra: Vectors, Matrices, and Least Squares*, Cambridge University Press 2018.
`https://web.stanford.edu/~boyd/vmls/vmls.pdf`