Math 220-002 Fall 2021 (Darij Grinberg): homework set 1
due date: Friday 2021-10-08 at noon on gradescope (
`https://www.gradescope.com/courses/313263`).
Please solve only **4 of the 6 exercises**.

Exercise 1. Define the logical connective UNLESS as follows: For any two propositions P and Q, we define "P UNLESS Q" to mean (NOT Q) $\Longrightarrow P$. (This is more or less the common-sense understanding of "unless".)

Prove (by comparing truth tables) that P UNLESS Q is actually equivalent to P OR Q.

Exercise 2. Let P, Q and R be three propositions. Prove the following equivalences by comparing truth tables:

(a) $(P \text{ OR } (Q \text{ AND } R)) \iff ((P \text{ OR } Q) \text{ AND } (P \text{ OR } R))$.

(b) $(P \text{ AND } (Q \text{ OR } R)) \iff ((P \text{ AND } Q) \text{ OR } (P \text{ AND } R))$.

(c) $(P \text{ XOR } (Q \text{ XOR } R)) \iff ((P \text{ XOR } Q) \text{ XOR } R)$.

(d) $((P \text{ OR } Q) \Longrightarrow R) \iff ((P \Longrightarrow R) \text{ AND } (Q \Longrightarrow R))$.

[**Hint:** You can save yourself some writing by making a common truth table with 8 rows and 11 columns for all 4 parts of the exercise.]

Exercise 3. Which of the following statements are true?

(a) $\forall x \in \mathbb{Z} : \ (2x \in \mathbb{Z})$.

(b) $\forall x \in \mathbb{Z} : \ \left(\dfrac{x}{2} \in \mathbb{Z}\right)$.

(c) $\forall x \in \mathbb{Z} : \ \left(x^3 \geq 0\right)$.

(d) $\forall x \in \mathbb{Z} : \ \left(x^4 \geq 0\right)$.

(e) $\forall x \in \mathbb{Z} : \ ((x \leq 3) \text{ OR } (x \geq 4))$.

(f) $\forall x \in \mathbb{Q} : \ ((x \leq 3) \text{ OR } (x \geq 4))$.

(g) $\forall x \in \mathbb{Q} : \ (\exists y \in \mathbb{Z} : \ y > x)$.

(h) $\forall x \in \mathbb{N} : \ (\exists y \in \mathbb{N} : \ (y > x \text{ AND } y \text{ is prime}))$.

In the cases of **(g)** and **(h)**, give a counterexample or a short intuitive justification (not a rigorous proof) in a sentence or two.

Exercise 4. **(a)** Find (and describe) a set S satisfying $\mathbb{Z} \subseteq S \subseteq \mathbb{Q}$ and $S \neq \mathbb{Z}$ and $S \neq \mathbb{Q}$.

(b) Let $S = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ be the set of all even integers, and let $T = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$ be the set of all integers that are multiples of 3. Let R denote the set $\{s + t \ | \ s \in S \text{ and } t \in T\}$; this means the set of all sums of an element of S with an element of T. This set R is a subset of $\mathbb{Z}$. Is R the whole set $\mathbb{Z}$ or a proper subset of $\mathbb{Z}$? (Justify your answer in a few sentences.)

Exercise 5. Let A, B and C be three sets. Prove the following equalities:

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(c) $A \bigtriangleup (B \bigtriangleup C) = (A \bigtriangleup B) \bigtriangleup C$.

[**Hint:** You can use the results of previous exercises.]

Exercise 6. Let S be a set, and let $P(x)$ and $Q(x)$ be two statements for each $x \in S$.

(a) Is it necessarily true that $(\exists x \in S : \ (P(x) \text{ OR } Q(x)))$ is equivalent to $(\exists x \in S : \ P(x)) \text{ OR } (\exists x \in S : \ Q(x))$?

(b) Is it necessarily true that $(\exists x \in S : \ (P(x) \text{ AND } Q(x)))$ is equivalent to $(\exists x \in S : \ P(x)) \text{ AND } (\exists x \in S : \ Q(x))$?

(c) Is it necessarily true that $(\exists x \in S : \ (P(x) \Longrightarrow Q(x)))$ is equivalent to $(\exists x \in S : \ P(x)) \Longrightarrow (\exists x \in S : \ Q(x))$?

(d) Is it necessarily true that $(\forall x \in S : \ (P(x) \text{ OR } Q(x)))$ is equivalent to $(\forall x \in S : \ P(x)) \text{ OR } (\forall x \in S : \ Q(x))$?

(e) Is it necessarily true that $(\forall x \in S : \ (P(x) \text{ AND } Q(x)))$ is equivalent to $(\forall x \in S : \ P(x)) \text{ AND } (\forall x \in S : \ Q(x))$?

(f) Is it necessarily true that $(\forall x \in S : \ (P(x) \Longrightarrow Q(x)))$ is equivalent to $(\forall x \in S : \ P(x)) \Longrightarrow (\forall x \in S : \ Q(x))$?

(By "necessarily true", we mean "true no matter what S, $P(x)$ and $Q(x)$ are". Almost everything becomes true in a sufficiently trivial example; we are wondering whether things are true in general. In each case, **give reasons or counterexamples**.)