**Math 220-002 Fall 2021 (Darij Grinberg): final exam**
due date: Sunday 2021-12-12 at noon on gradescope
( `https://www.gradescope.com/courses/313263` ).
Please solve only **4 of the 6 exercises**.

**Exercise 1.** Let $n \in \mathbb{N}$. Prove that

$$\underbrace{1+2+\cdots+n}_{=\sum_{k=1}^{n} k} = \underbrace{n^2-(n-1)^2+(n-2)^2-(n-3)^2 \pm \cdots + (-1)^{n-1}1^2}_{=\sum_{k=1}^{n}(-1)^{n-k}k^2}.$$

Recall the notation $\prod_{k=i}^{j} a_k$ for a product $a_i a_{i+1} a_{i+2} \cdots a_j$. (If $i > j$, then this product is empty and understood to equal 1. If $i = j$, then this product has only one factor and equals $a_i$.)

**Exercise 2.** Prove the following:

**(a)** For each $n \in \mathbb{N}$, we have

$$\prod_{k=0}^{n} k! = \prod_{k=1}^{n} k! = \prod_{k=1}^{n} k^{n-k+1}.$$

**(b)** For each $n \in \mathbb{N}$, we have

$$\prod_{k=1}^{n} k^{k-1} = \prod_{k=1}^{n} \frac{n!}{k!}.$$

**Exercise 3.** Let $m$ be a positive integer, and let $m_d m_{d-1} \cdots m_0$ be its decimal representation, so that $m_0, m_1, \ldots, m_d$ are digits satisfying

$$m = m_d \cdot 10^d + m_{d-1} \cdot 10^{d-1} + \cdots + m_0 \cdot 10^0.$$

Let $a$ be the alternating sum of digits of $m$; this is defined by

$$a := m_0 - m_1 + m_2 - m_3 \pm \cdots + (-1)^d m_d = \sum_{k=0}^{d} (-1)^k m_k.$$

Prove that $11 \mid m$ if and only if $11 \mid a$. (This is the classical divisibility test for divisibility by 11.)

**Exercise 4.** Let $n$ be an integer such that $n > 1$ but $n$ is not a prime. Let $d$ be the smallest divisor of $n$ that is larger than 1. Prove that $d \leq \sqrt{n}$.

(You can use standard properties of inequalities – e.g., the equivalence $(u \leq v) \iff (u^2 \leq v^2)$ when $u$ and $v$ are positive.)

**Exercise 5.** Let $p$ be a prime.

**(a)** Prove that $(a+1)^p \equiv a^p + 1 \mod p$ for each integer $a$.

**(b)** Prove that $a^p \equiv a \mod p$ for each $a \in \mathbb{N}$.

**(c)** Prove that $a^p \equiv a \mod p$ for each integer $a$.

[**Hint:** The binomial formula and a certain property of $\dbinom{p}{k}$ can help.]

**Exercise 6.** Let $X$, $Y$ and $Z$ be three sets, and $f : Y \to Z$ and $g : X \to Y$ be two maps. Which of the following are true?

**(a)** If $f$ and $g$ are injective, then $f \circ g$ is injective.

**(b)** If $f \circ g$ is injective, then $f$ is injective.

**(c)** If $f \circ g$ is injective, then $g$ is injective.

**(d)** If $f$ and $g$ are surjective, then $f \circ g$ is surjective.

**(e)** If $f \circ g$ is surjective, then $f$ is surjective.

**(f)** If $f \circ g$ is surjective, then $g$ is surjective.

**(g)** If $f$ and $g$ are bijective, then $f \circ g$ is bijective.

**(h)** If $f \circ g$ is bijective, then $f$ is bijective.

**(i)** If $f \circ g$ is bijective, then $g$ is bijective.

**(j)** Pick **one** of the nine claims above and justify your answer (with a proof if the answer is "yes" and a counterexample if the answer is "no").

[Other than this, justifications are **not** required in this exercise!]