Math 504: Advanced Linear Algebra

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Contents

Math 504 Lecture 17

1. Hermitian matrices (cont'd)

1.1. Rayleigh quotients (cont'd)

Recall:

> **Definition 1.1.1.** Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix, and $x \in \mathbb{C}^n$ be a nonzero vector. Then, the **Rayleigh quotient** for A and x is defined to be the real number
>
> $$R(A, x) := \frac{\langle Ax, x \rangle}{\langle x, x \rangle} = \frac{x^* A x}{x^* x} = \frac{x^* A x}{||x||^2} = y^* A y,$$
>
> where $y = \dfrac{x}{||x||}$.

We need to show:

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Theorem 1.1.2 (Courant–Fisher theorem)**.** Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A, with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then, for each $k \in [n]$, we have

$$\lambda_k = \min_{\substack{S\subseteq\mathbb{C}^n \text{ is a subspace;} \\ \dim S=k}} \ \max_{\substack{x\in S; \\ x\neq 0}} R(A,x)$$

and

$$\lambda_k = \max_{\substack{S\subseteq\mathbb{C}^n \text{ is a subspace;} \\ \dim S=n-k+1}} \ \min_{\substack{x\in S; \\ x\neq 0}} R(A,x).$$

To prove this theorem, we will use some elementary facts about subspaces of finite-dimensional vector spaces. We begin by recalling the following definition:

Definition 1.1.3. Let S_1 and S_2 be two subspaces of a vector space V. Then,

$$S_1 + S_2 := \{s_1 + s_2 \mid s_1 \in S_1 \text{ and } s_2 \in S_2\}.$$

This is again a subspace of V. (This is the smallest subspace of V that contains both S_1 and S_2 as subspaces.)

Proposition 1.1.4. Let $\mathbb{F}$ be a field. Let V be a finite-dimensional $\mathbb{F}$-vector space. Let S_1 and S_2 be two subspaces of V. Then,

$$\dim(S_1 \cap S_2) + \dim(S_1 + S_2) = \dim S_1 + \dim S_2.$$

Proof. Pick any basis $(x_1, x_2, \ldots, x_k)$ of the vector space $S_1 \cap S_2$.

Then, $(x_1, x_2, \ldots, x_k)$ is a linearly independent list of vectors in S_1. Thus, we can extend it to a basis of S_1 by inserting some new vectors $y_1, y_2, \ldots, y_p$. Thus,

$$(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_p) \text{ is a basis of } S_1.$$

On the other hand, $(x_1, x_2, \ldots, x_k)$ is a linearly independent list of vectors in S_2. Thus, we can extend it to a basis of S_2 by inserting some new vectors $z_1, z_2, \ldots, z_q$. Thus,

$$(x_1, x_2, \ldots, x_k, z_1, z_2, \ldots, z_q) \text{ is a basis of } S_2.$$

The above three bases yield $\dim(S_1 \cap S_2) = k$ and $\dim S_1 = k + p$ and $\dim S_2 = k + q$.

Now, we claim that

$$\mathbf{w} := (x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_p, z_1, z_2, \ldots, z_q) \text{ is a basis of } S_1 + S_2.$$

Once this is proved, we will conclude that $\dim(S_1 + S_2) = k + p + q$, and the proposition will follow by a simple computation ($k + (k + p + q) = (k + p) + (k + q)$).

So let us prove our claim. To prove that $\mathbf{w}$ is a basis of $S_1 + S_2$, we need to check the following two statements:

1. The list $\mathbf{w}$ is linearly independent.

2. The list $\mathbf{w}$ spans $S_1 + S_2$.

Proving statement 2 is easy: Any element of $S_1 + S_2$ is an element of S_1 plus an element of S_2, and thus can be written as

$$\begin{aligned}
&\left(\text{a linear combination of } x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_p\right) \\
&\qquad + \left(\text{a linear combination of } x_1, x_2, \ldots, x_k, z_1, z_2, \ldots, z_q\right) \\
&= \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k + \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_p y_p \\
&\qquad + \mu_1 x_1 + \mu_2 x_2 + \cdots + \mu_k x_k + \beta_1 z_1 + \beta_2 z_2 + \cdots + \beta_q z_q \\
&= \left(\lambda_1 + \mu_1\right) x_1 + \left(\lambda_2 + \mu_2\right) x_2 + \cdots + \left(\lambda_k + \mu_k\right) x_k \\
&\qquad + \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_p y_p + \beta_1 z_1 + \beta_2 z_2 + \cdots + \beta_q z_q \\
&= \left(\text{a linear combination of } x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_p, z_1, z_2, \ldots, z_q\right);
\end{aligned}$$

thus it belongs to the span of $\mathbf{w}$.

Let us now prove statement 1. We need to show that $\mathbf{w}$ is linearly independent. So let us assume that

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k + \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_p y_p + \beta_1 z_1 + \beta_2 z_2 + \cdots + \beta_q z_q = 0$$

for some coefficients $\lambda_m, \alpha_i, \beta_j$ that are not all equal to 0. We want a contradiction.

Let

$$v := \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k + \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_p y_p.$$

Then,

$$\begin{aligned}
v &= -\left(\beta_1 z_1 + \beta_2 z_2 + \cdots + \beta_q z_q\right) \qquad \text{(by the above equation)} \\
&\in S_2 \qquad \left(\text{since the } z_j\text{'s lie in } S_2\right).
\end{aligned}$$

On the other hand, the definition of v yields $v \in S_1$ (since the x_m's and the y_i's lie in S_1). Thus, v lies in both S_1 and S_2. This entails that $v \in S_1 \cap S_2$. Since $(x_1, x_2, \ldots, x_k)$ is a basis of $S_1 \cap S_2$, this entails that

$$v = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_k x_k \qquad \text{for some } \xi_1, \xi_2, \ldots, \xi_k \in \mathbb{F}.$$

Comparing this with

$$v = -\left(\beta_1 z_1 + \beta_2 z_2 + \cdots + \beta_q z_q\right),$$

we obtain

$$\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_k x_k = -\left(\beta_1 z_1 + \beta_2 z_2 + \cdots + \beta_q z_q\right).$$

In other words,

$$\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_k x_k + \beta_1 z_1 + \beta_2 z_2 + \cdots + \beta_q z_q = 0.$$

Since the list $(x_1, x_2, \ldots, x_k, z_1, z_2, \ldots, z_q)$ is linearly independent (being a basis of S_2), this entails that all coefficients ξ_m and β_j are 0. Thus, $v = 0$ (since $v = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_k x_k$). Now, recalling that

$$v = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k + \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_p y_p,$$

we obtain

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k + \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_p y_p = 0.$$

Since the list $(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_p)$ is linearly independent (being a basis of S_1), this entails that all coefficients λ_m and α_i are 0.

Now we know that all λ_m and α_i and β_j are 0, which contradicts our assumption that some of them are nonzero. This completes the proof of Statement 1.

As we said, we now conclude that $\dim(S_1 + S_2) = k + p + q$, so that

$$\begin{aligned}
\underbrace{\dim(S_1 \cap S_2)}_{=k} + \underbrace{\dim(S_1 + S_2)}_{=k+p+q} &= k + (k + p + q) \\
&= \underbrace{(k+p)}_{=\dim S_1} + \underbrace{(k+q)}_{=\dim S_2} \\
&= \dim S_1 + \dim S_2.
\end{aligned}$$

□

Remark 1.1.5. A well-known fact in elementary set theory says that if A_1 and A_2 are two finite sets, then

$$|A_1 \cap A_2| + |A_1 \cup A_2| = |A_1| + |A_2|.$$

The above theorem is an analogue of this fact for vector spaces (noticing that the sum $S_1 + S_2$ is a vector-space analogue of the union).

Note, however, that the "next level" of the above formula has no vector space analogue. We do have

$$\begin{aligned}
&|A_1 \cup A_2 \cup A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\
&= |A_1| + |A_2| + |A_3| + |A_1 \cap A_2 \cap A_3|
\end{aligned}$$

for any three finite sets A_1, A_2, A_3, but no such relation holds for three subspaces of a vector space.

Corollary 1.1.6. Let $\mathbb{F}$ be a field, and let $n \in \mathbb{N}$. Let V be an n-dimensional $\mathbb{F}$-vector space. Let $S_1, S_2, \ldots, S_k$ be subspaces of V (with $k \geq 1$). Let

$$\delta := \dim(S_1) + \dim(S_2) + \cdots + \dim(S_k) - (k-1)\,n.$$

(a) Then, $\dim(S_1 \cap S_2 \cap \cdots \cap S_k) \geq \delta$.

(b) If $\mathbb{F} = \mathbb{C}$ and $V = \mathbb{C}^n$ and $\delta > 0$, then there exists a vector $x \in S_1 \cap S_2 \cap \cdots \cap S_k$ with $||x|| = 1$.

Proof. **(a)** We induct on k. The *base case* ($k = 1$) is obvious (since $\dim(S_1 \cap S_2 \cap \cdots \cap S_k) = \dim(S_1) = \delta$ in this case).

Induction step: Suppose the statement holds for some k. Now consider $k+1$ subspaces $S_1, S_2, \ldots, S_{k+1}$ of V, and let

$$\delta_{k+1} := \dim(S_1) + \dim(S_2) + \cdots + \dim(S_{k+1}) - kn.$$

We want to prove that $\dim(S_1 \cap S_2 \cap \cdots \cap S_k \cap S_{k+1}) \geq \delta_{k+1}$.

Then,

$$\begin{aligned} &\dim(S_1 \cap S_2 \cap \cdots \cap S_k \cap S_{k+1}) \\ &= \dim(S_1 \cap S_2 \cap \cdots \cap S_{k-1} \cap (S_k \cap S_{k+1})). \end{aligned}$$

Now, set

$$\delta_k := \dim(S_1) + \dim(S_2) + \cdots + \dim(S_{k-1}) + \dim(S_k \cap S_{k+1}) - (k-1)n.$$

By the induction hypothesis, we have

$$\dim(S_1 \cap S_2 \cap \cdots \cap S_{k-1} \cap (S_k \cap S_{k+1})) \geq \delta_k.$$

What remains is to show that $\delta_k \geq \delta_{k+1}$. Equivalently, we need to show that

$$\dim(S_k \cap S_{k+1}) - (k-1)n \geq \dim(S_k) + \dim(S_{k+1}) - kn.$$

In other words, we need to show that

$$\dim(S_k \cap S_{k+1}) + n \geq \dim(S_k) + \dim(S_{k+1}).$$

However, $S_k + S_{k+1}$ is a subspace of V, so its dimension is $\dim(S_k + S_{k+1}) \leq \dim V = n$. Therefore,

$$\dim(S_k \cap S_{k+1}) + \underbrace{n}_{\geq \dim(S_k + S_{k+1})} \geq \dim(S_k \cap S_{k+1}) + \dim(S_k + S_{k+1}) = \dim(S_k) + \dim(S_{k+1})$$

(by the previous proposition). So the induction step is complete, and part **(a)** of the corollary is proved.

(b) Assume that $\mathbb{F} = \mathbb{C}$ and $V = \mathbb{C}^n$ and $\delta > 0$. Then, part **(a)** yields

$$\dim(S_1 \cap S_2 \cap \cdots \cap S_k) \geq \delta > 0.$$

Thus, the subspace $S_1 \cap S_2 \cap \cdots \cap S_k$ is not just $\{0\}$. Therefore, it contains a nonzero vector. Scaling this vector by the reciprocal of its length, we obtain a vector of length 1. This proves part **(b)**. $\square$

Now, we get to the proof of the Courant–Fisher theorem:

Proof of the Courant–Fisher theorem. The spectral theorem says that $A = UDU^*$ for some unitary U and some real diagonal matrix D. Consider these U and D. The columns of U form an orthonormal basis of $\mathbb{C}^n$ (since U is unitary); let $(u_1, u_2, \ldots, u_n)$ be this basis. Then, $u_1, u_2, \ldots, u_n$ are eigenvectors of A. We WLOG assume that the corresponding eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$ (otherwise, permute the diagonal entries of D and correspondingly permute the columns of U).

Let $k \in [n]$.

Let S be a vector subspace of $\mathbb{C}^n$ with $\dim S = k$. Let $S' = \operatorname{span}(u_k, u_{k+1}, \ldots, u_n)$. Then, by the proposition, we have

$$\dim\left(S \cap S'\right) + \dim\left(S + S'\right) = \underbrace{\dim S}_{=k} + \underbrace{\dim S'}_{=n-k+1} = n + 1 > n \geq \dim\left(S + S'\right)$$

(since $S + S'$ is a subspace of $\mathbb{C}^n$). Subtracting $\dim(S + S')$ from this inequality, we obtain $\dim(S \cap S') > 0$. Thus, $S \cap S'$ contains a nonzero vector. Thus, $\sup\limits_{x \in S \cap S';\ x \neq 0} R(A, x)$ and $\inf\limits_{x \in S \cap S';\ x \neq 0} R(A, x)$ are well-defined.

Now,

$$\sup_{\substack{x \in S;\\ x \neq 0}} R(A, x) \geq \sup_{\substack{x \in S \cap S';\\ x \neq 0}} R(A, x) \geq \inf_{\substack{x \in S \cap S';\\ x \neq 0}} R(A, x) \geq \inf_{\substack{x \in S';\\ x \neq 0}} R(A, x).$$

However, I claim that $\inf\limits_{x \in S';\ x \neq 0} R(A, x) = \lambda_k$. Indeed, any $x \in S'$ is a linear combination $\alpha_k u_k + \alpha_{k+1} u_{k+1} + \cdots + \alpha_n u_n$ and therefore satisfies

$$\begin{aligned}
\langle Ax, x \rangle &= \langle A\left(\alpha_k u_k + \alpha_{k+1} u_{k+1} + \cdots + \alpha_n u_n\right),\ \alpha_k u_k + \alpha_{k+1} u_{k+1} + \cdots + \alpha_n u_n \rangle \\
&= \langle \alpha_k A u_k + \alpha_{k+1} A u_{k+1} + \cdots + \alpha_n A u_n,\ \alpha_k u_k + \alpha_{k+1} u_{k+1} + \cdots + \alpha_n u_n \rangle \\
&= \langle \alpha_k \lambda_k u_k + \alpha_{k+1} \lambda_{k+1} u_{k+1} + \cdots + \alpha_n \lambda_n u_n,\ \alpha_k u_k + \alpha_{k+1} u_{k+1} + \cdots + \alpha_n u_n \rangle \\
&= \sum_{i=k}^{n} \underbrace{\alpha_i \overline{\alpha_i}}_{=|\alpha_i|^2} \lambda_i \qquad \text{(since } u_k, u_{k+1}, \ldots, u_n \text{ are orthonormal)} \\
&= \sum_{i=k}^{n} |\alpha_i|^2 \underbrace{\lambda_i}_{\substack{\geq \lambda_k \\ \text{(since } \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n)}} \geq \lambda_k \underbrace{\sum_{i=k}^{n} |\alpha_i|^2}_{=\langle x, x \rangle} = \lambda_k \langle x, x \rangle
\end{aligned}$$

and thus $R(A, x) = \dfrac{\langle Ax, x \rangle}{\langle x, x \rangle} \geq \lambda_k$. So, altogether, we find

$$\sup_{\substack{x \in S;\\ x \neq 0}} R(A, x) \geq \lambda_k.$$

Furthermore, this supremum is a maximum, because

$$\begin{aligned}
\sup_{\substack{x\in S;\\ x\neq 0}} R(A,x) &= \sup_{\substack{y\in S;\\ \|y\|=1}} R(A,y) && \left(\text{since } R(A,x)=R(A,y) \text{ where } y=\frac{x}{\|x\|}\right)\\
&= \max_{\substack{y\in S;\\ \|y\|=1}} R(A,y) && \left(\begin{array}{c}\text{since the set of all } y\in S \text{ satisfying } \|y\|=1\\ \text{is compact, and since a continuous function}\\ \text{on a compact set always has a maximum}\end{array}\right)\\
&= \max_{\substack{x\in S;\\ x\neq 0}} R(A,x).
\end{aligned}$$

So we conclude

$$\max_{\substack{x\in S;\\ x\neq 0}} R(A,x) = \sup_{\substack{x\in S;\\ x\neq 0}} R(A,x) \geq \lambda_k.$$

Forget that we fixed S. We thus have shown that if S is any k-dimensional subspace of $\mathbb{C}^n$, then $\max\limits_{\substack{x\in S;\\ x\neq 0}} R(A,x)$ exists and satisfies

$$\max_{\substack{x\in S;\\ x\neq 0}} R(A,x) \geq \lambda_k.$$

However, by choosing S appropriately, we can achieve equality here; indeed, we have to choose $S=\operatorname{span}(u_1,u_2,\ldots,u_k)$ for this. (Why? Because each $x\in\operatorname{span}(u_1,u_2,\ldots,u_k)$ can easily be seen to satisfy $\langle Ax,x\rangle \leq \lambda_k\langle x,x\rangle$ by a similar argument to the one we used above.)

So $\max\limits_{\substack{x\in S;\\ x\neq 0}} R(A,x)$ is $\geq \lambda_k$ for each S, but is $=\lambda_k$ for a certain S. Therefore, λ_k is the smallest possible value of $\max\limits_{\substack{x\in S;\\ x\neq 0}} R(A,x)$. In other words,

$$\lambda_k = \min_{\substack{S\subseteq\mathbb{C}^n \text{ is a subspace;}\\ \dim S=k}} \max_{\substack{x\in S;\\ x\neq 0}} R(A,x).$$

It remains to prove the other part of the theorem – i.e., the equality

$$\lambda_k = \max_{\substack{S\subseteq\mathbb{C}^n \text{ is a subspace;}\\ \dim S=n-k+1}} \min_{\substack{x\in S;\\ x\neq 0}} R(A,x).$$

One way to prove this is by arguing similarly to the above proof. Alternatively, we can simply apply the already proved equality

$$\lambda_k = \min_{\substack{S\subseteq\mathbb{C}^n \text{ is a subspace;}\\ \dim S=k}} \max_{\substack{x\in S;\\ x\neq 0}} R(A,x)$$

to $-A$ instead of A, which is a Hermitian matrix with eigenvalues

$$-\lambda_n \leq -\lambda_{n-1} \leq \cdots \leq -\lambda_1.$$

Keep in mind that $-\lambda_k$ is not the k-th smallest eigenvalue of $-A$, but it is the k-th largest eigenvalue of $-A$, and thus the $(n-k+1)$-st smallest eigenvalue of $-A$. Thus, we have to apply the equality to $-A$ and $n-k+1$ instead of A and k. Taking negatives turns minima into maxima and vice versa. $\square$

The Courant–Fisher theorem can be used to connect the eigenvalues of $A+B$ with the eigenvalues of A and B.

Theorem 1.1.7 (Weyl's theorem)**.** Let A and B be two Hermitian matrices in $\mathbb{C}^{n\times n}$. Let $i \in [n]$ and $j \in \{0,1,\ldots,n-i\}$.

(a) Then,

$$\lambda_i (A+B) \leq \lambda_{i+j}(A) + \lambda_{n-j}(B).$$

Here, $\lambda_k(C)$ means the k-th smallest eigenvalue of a Hermitian matrix C.

Moreover, this inequality becomes an equality if and only if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying

$$Ax = \lambda_{i+j}(A)\, x, \qquad Bx = \lambda_{n-j}(B)\, x, \qquad (A+B)\, x = \lambda_i (A+B)\, x$$

(at the same time).

(b) Furthermore,

$$\lambda_{i-k+1}(A) + \lambda_k(B) \leq \lambda_i(A+B) \qquad \text{for any } k \in [i].$$

Proof. Let $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ and $(z_1, z_2, \ldots, z_n)$ be three orthonormal bases of $\mathbb{C}^n$ with

$$Ax_i = \lambda_i(A)\, x_i, \qquad By_i = \lambda_i(B)\, y_i, \qquad (A+B)\, z_i = \lambda_i(A+B)\, z_i$$

for all $i \in [n]$. (As above, we can find such bases by using the spectral decompositions of A, B and $A+B$.)

Let

$$\begin{aligned} S_1 &= \operatorname{span}\left(x_1, x_2, \ldots, x_{i+j}\right); \\ S_2 &= \operatorname{span}\left(y_1, y_2, \ldots, y_{n-j}\right); \\ S_3 &= \operatorname{span}\left(z_i, z_{i+1}, \ldots, z_n\right). \end{aligned}$$

Then,

$$\delta := \underbrace{\dim(S_1)}_{=i+j} + \underbrace{\dim(S_2)}_{=n-j} + \underbrace{\dim(S_3)}_{=n-i+1} - 2n = 1 > 0.$$

Hence, part **(b)** of the corollary yields that there is a length-1 vector x in $S_1 \cap S_2 \cap S_3$. This x satisfies

$$\lambda_i (A+B) \leq \langle (A+B)x, x \rangle = \langle Ax + Bx, x \rangle = \underbrace{\langle Ax, x \rangle}_{\leq \lambda_{i+j}(A)} + \underbrace{\langle Bx, x \rangle}_{\leq \lambda_{n-j}(B)}$$
$$\leq \lambda_{i+j}(A) + \lambda_{n-j}(B).$$

This proves **(a)**.

We leave **(b)** and **(c)** to the reader. □