Math 504: Advanced Linear Algebra

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Contents

Math 504 Lecture 13

1. Jordan canonical (aka normal) form (cont'd)

1.1. The companion matrix

For each $n \times n$-matrix A, we have defined its characteristic polynomial p_A and its minimal polynomial q_A. What variety of polynomials do we get this way? Do all characteristic polynomials share some property, or can any monic polynomial be a characteristic polynomial? The same question for minimal polynomials?

Poll:

1. Any monic polynomial can be a characteristic polynomial.
2. Characteristic polynomials have some special property among monic polynomials.

The answer is 1.

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Definition 1.1.1. Let $\mathbb{F}$ be a field, and let $n \in \mathbb{N}$.

Let $p(t) = t^n + p_{n-1}t^{n-1} + p_{n-2}t^{n-2} + \cdots + p_1 t^1 + p_0 t^0$ be a monic polynomial of degree n with coefficients in $\mathbb{F}$. Then, the **companion matrix** of $p(t)$ is defined to be the matrix

$$C_p := \begin{pmatrix} 0 & & & \cdots & & -p_0 \\ 1 & 0 & & \cdots & & -p_1 \\ & 1 & 0 & \cdots & & -p_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & \cdots & 0 & -p_{n-2} \\ & & & \cdots & 1 & -p_{n-1} \end{pmatrix} \in \mathbb{F}^{n\times n}.$$

This is the $n \times n$-matrix whose first $n-1$ columns are the standard basis vectors $e_2, e_3, \ldots, e_n$, and whose last column is $(-p_0, -p_1, \ldots, -p_{n-1})^T$.

Proposition 1.1.2. For any monic polynomial $p(t)$, we have

$$p_{C_p}(t) = q_{C_p}(t) = p(t).$$

(The two p's in "p_{C_p}" stand for different things: The first stands for "characteristic polynomial", while the second is the $p(t)$ we have given.)

Proof. Let us first show that $p_{C_p}(t) = p(t)$. To do so, we induct on n. Recall that

$$\begin{aligned} p_{C_p}(t) &= \det\left(tI_n - C_p\right) \\ &= \det \begin{pmatrix} t & & & \cdots & & p_0 \\ -1 & t & & \cdots & & p_1 \\ & -1 & t & \cdots & & p_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & \cdots & t & p_{n-2} \\ & & & \cdots & -1 & t+p_{n-1} \end{pmatrix}. \end{aligned}$$

We compute this determinant by Laplace expansion along the first row:

$$
\begin{aligned}
&\det\begin{pmatrix}
t & & & \cdots & & p_0 \\
-1 & t & & \cdots & & p_1 \\
 & -1 & t & \cdots & & p_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & \cdots & t & p_{n-2} \\
 & & & \cdots & -1 & t+p_{n-1}
\end{pmatrix} \\
&= t\ \underbrace{\det\begin{pmatrix}
t & & \cdots & & p_1 \\
-1 & t & \cdots & & p_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
 & & \cdots & t & p_{n-2} \\
 & & \cdots & -1 & t+p_{n-1}
\end{pmatrix}}_{\substack{=t^{n-1}+p_{n-1}t^{n-2}+\cdots+p_2t^1+p_1t^0 \\ \text{(by the induction hypothesis)}}}
+ (-1)^{n+1} p_0\ \underbrace{\det\begin{pmatrix}
-1 & t & & \cdots & \\
 & -1 & t & \cdots & \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
 & & & \cdots & t \\
 & & & \cdots & -1
\end{pmatrix}}_{\substack{=(-1)^{n-1} \\ \text{(since this matrix is upper-triangular} \\ \text{of size } n-1\text{)}}} \\
&= t\left(t^{n-1}+p_{n-1}t^{n-2}+\cdots+p_2t^1+p_1t^0\right) + \underbrace{(-1)^{n+1} p_0 (-1)^{n-1}}_{=p_0} \\
&= t\left(t^{n-1}+p_{n-1}t^{n-2}+\cdots+p_2t^1+p_1t^0\right) + p_0 \\
&= t^n + p_{n-1}t^{n-1} + p_2t^2 + p_1t^1 + p_0 = p(t).
\end{aligned}
$$

Thus, $p_{C_p}(t) = p(t)$ is proved.

Now, let us show that $q_{C_p}(t) = p(t)$. Indeed, both $q_{C_p}(t)$ and $p(t)$ are monic polynomials, and we know from last lecture that $q_{C_p}(t) \mid p_{C_p}(t) = p(t)$. Hence, if $q_{C_p}(t) \neq p(t)$, then $q_{C_p}(t)$ is a proper divisor of $p(t)$, thus has degree $< n$ (since $p(t)$ has degree n). So we just need to rule out the possibility that $q_{C_p}(t)$ has degree $< n$.

Indeed, assume (for the sake of contradiction) that $q_{C_p}(t)$ has degree $< n$. Thus, $q_{C_p}(t) = a_kt^k + a_{k-1}t^{k-1} + \cdots + a_0t^0$ with $k < n$ and $a_k = 1$ (since q_{C_p} is monic of degree $< n$). However, the definition of q_{C_p} yields $q_{C_p}(C_p) = 0$. In other words,

$$a_kC_p^k + a_{k-1}C_p^{k-1} + \cdots + a_0C_p^0 = 0.$$

However, let us look at what C_p does to the standard basis vector $e_1 = (1,0,0,0,\ldots,0)^T$. We have

$$
\begin{aligned}
C_p^0e_1 &= e_1; \\
C_p^1e_1 &= C_pe_1 = e_2; \\
C_p^2e_1 &= C_pe_2 = e_3; \\
&\ldots; \\
C_p^{n-1}e_1 &= e_n.
\end{aligned}
$$

Thus, applying our equality

$$a_k C_p^k + a_{k-1} C_p^{k-1} + \cdots + a_0 C_p^0 = 0$$

to e_1, we obtain

$$a_k e_{k+1} + a_{k-1} e_k + \cdots + a_0 e_1 = 0 \qquad (\text{since } k < n).$$

But this is absurd, since $e_1, e_2, \ldots, e_n$ are linearly independent. So we found a contradiction, and thus we conclude that $q_{C_p}(t)$ has degree $\geq n$. So, by the above, we obtain $q_{C_p}(t) = p(t)$. □

Remark 1.1.3. For algebraists: The companion matrix C_p has a natural meaning. To wit, consider the quotient ring $\mathbb{F}[t] / (p(t))$ as an n-dimensional $\mathbb{F}$-vector space with basis $\left(\overline{t^0}, \overline{t^1}, \ldots, \overline{t^{n-1}}\right)$. Then, the companion matrix C_p represents the endomorphism "multiply by t" (that is, the endomorphism that sends each $\overline{f(t)}$ to $\overline{t \cdot f(t)}$) in this basis.

1.2. The Jordan–Chevalley decomposition

Recall that:

- A matrix $A \in \mathbb{C}^{n \times n}$ is said to be **diagonalizable** if it is similar to a diagonal matrix.
- A matrix $A \in \mathbb{C}^{n \times n}$ is said to be **nilpotent** if some power of it is the zero matrix (i.e., if $A^k = 0$ for some $k \in \mathbb{N}$). Actually (this will be a HW problem), for an $n \times n$-matrix A to be nilpotent, it is necessary and sufficient that $A^n = 0$.

Theorem 1.2.1 (Jordan–Chevalley decomposition)**.** Let $A \in \mathbb{C}^{n \times n}$ be an $n \times n$-matrix.

(a) Then, there exists a unique pair (D, N) consisting of

- a diagonalizable matrix $D \in \mathbb{C}^{n \times n}$ and
- a nilpotent matrix $N \in \mathbb{C}^{n \times n}$

such that $DN = ND$ and $A = D + N$.

(b) Both D and N in this pair can be written as polynomials in A. In other words, there exist two polynomials $f, g \in \mathbb{C}[t]$ such that $D = f(A)$ and $N = g(A)$.

The pair (D, N) in this theorem is known as the **Jordan–Chevalley decomposition** (or the **Dunford decomposition**) of A.

Partial proof. We will show the following two claims:

Claim 1: There exists a pair (D, N) as in part **(a)** of the theorem.

Claim 2: The D and N in this particular pair can be written as polynomials in A.

To prove both Claims 1 and 2, we can WLOG assume that A is a Jordan matrix. Indeed, if $A = SJS^{-1}$ for some invertible S, and if (D', N') is a Jordan–Chevalley decomposition of J, then $\left(SD'S^{-1}, SN'S^{-1}\right)$ is a Jordan–Chevalley decomposition of A. Conjugation of matrices preserves all the properties we need (such as being a polynomial, commuting, etc.), so we only need to prove the claims for J.

So we WLOG assume that A is a Jordan matrix. Thus,

$$A = \begin{pmatrix} J_{k_1}(\lambda_1) & & & \\ & J_{k_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{k_p}(\lambda_p) \end{pmatrix}$$

for some $\lambda_1, \lambda_2, \ldots, \lambda_p$ and some $k_1, k_2, \ldots, k_p$. Now, we want to decompose this A as $D + N$ with D diagonal and N nilpotent and $DN = ND$. We do this by setting

$$D := \begin{pmatrix} \lambda_1 I_{k_1} & & & \\ & \lambda_2 I_{k_2} & & \\ & & \ddots & \\ & & & \lambda_p I_{k_p} \end{pmatrix} \qquad \text{and} \qquad N := \begin{pmatrix} J_{k_1}(0) & & & \\ & J_{k_2}(0) & & \\ & & \ddots & \\ & & & J_{k_p}(0) \end{pmatrix}.$$

It is easy to check that $A = D + N$ and $DN = ND$ (since block-diagonal matrices can be multiplied block by block). Clearly, D is diagonalizable (since D is diagonal) and N is nilpotent (since N is strictly upper-triangular). Thus, Claim 1 is proved.

Now, we need to prove Claim 2 – i.e., we need to prove that D and N can be written as polynomials in A. Since $A = D + N$, it suffices to show this for D (since $N = A - D$).

Our above construction of D shows that D is simply A with its non-diagonal entries removed. Let $\mu_1, \mu_2, \ldots, \mu_m$ be the **distinct** eigenvalues (i.e., diagonal entries) of A. For each $i \in [m]$, let ℓ_i be the size of the largest Jordan block of A at eigenvalue μ_i. (Thus, the minimal polynomial of A is $\prod_{i=1}^{m} (t - \mu_i)^{\ell_i}$.)

Now, define the polynomial

$$f(t) := \sum_{i=1}^{m} \mu_i \prod_{j \neq i} \left(\frac{t - \mu_j}{\mu_i - \mu_j} \right)^{\ell_j} \in \mathbb{C}[t],$$

where the product sign $\prod_{j \neq i}$ means a product over all $j \in [m]$ except for $j = i$. We claim that $f(A) = D$. In other words, we claim that applying f to A has the effect of cleaning out all off-diagonal entries (while the diagonal entries remain as they are).

To prove this claim, we recall that

$$A = \begin{pmatrix} J_{k_1}(\lambda_1) & & & \\ & J_{k_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{k_p}(\lambda_p) \end{pmatrix}.$$

Hence,

$$f(A) = \begin{pmatrix} f(J_{k_1}(\lambda_1)) & & & \\ & f(J_{k_2}(\lambda_2)) & & \\ & & \ddots & \\ & & & f\left(J_{k_p}(\lambda_p)\right) \end{pmatrix}.$$

So we need to show that

$$f(J_{k_u}(\lambda_u)) = \lambda_u I_{k_u} \qquad \text{for each } u \in [p].$$

To prove this, we fix a $u \in [p]$, and we set $B := J_{k_u}(\lambda_u)$. Thus, $B = J_k(\mu_v)$ for some $v \in [p]$ and some positive $k \leq \ell_v$.

Substituting B for t into

$$f(t) := \sum_{i=1}^{m} \mu_i \prod_{j \neq i} \left(\frac{t - \mu_j}{\mu_i - \mu_j} \right)^{\ell_j},$$

we obtain

$$f(B) = \sum_{i=1}^{m} \mu_i \prod_{j \neq i} \left(\frac{B - \mu_j I}{\mu_i - \mu_j} \right)^{\ell_j}.$$

We take a closer look at the addends of the sum. For each $i \in [m]$ that is distinct from v, the product $\prod\limits_{j \neq i} \left(\dfrac{B - \mu_j I}{\mu_i - \mu_j} \right)^{\ell_j}$ contains a factor $\left(\dfrac{B - \mu_v I}{\mu_i - \mu_v} \right)^{\ell_v} = 0$, because the $k \times k$-matrix $\dfrac{B - \mu_v I}{\mu_i - \mu_v}$ is strictly upper-triangular and $k \leq \ell_v$. So the entire addend $\mu_i \prod\limits_{j \neq i} \left(\dfrac{B - \mu_j I}{\mu_i - \mu_j} \right)^{\ell_j}$ is 0 whenever i is distinct from v. Thus, all these addends disappear except for the addend for $i = v$. So the above formula for $f(B)$ simplifies to

$$f(B) = \mu_v \prod_{j \neq v} \left(\frac{B - \mu_j I}{\mu_v - \mu_j} \right)^{\ell_j}.$$

This should be $\mu_v I$.

I'm just seeing this isn't exactly the case. TODO: Fix this.

[Alternatively, use the Chinese Remainder Theorem for polynomials to find a polynomial f that satisfies $f\left(J_{k_u}\left(\lambda_u\right)\right) = \lambda_u I_{k_u}$. This polynomial f should satisfy $f \equiv (t-\lambda_u)^{k_u} + \lambda_u$ for each u.] $\square$

1.3. The real Jordan canonical form

Given a matrix $A \in \mathbb{R}^{n\times n}$ with real entries, its Jordan canonical form doesn't necessarily have real entries. Indeed, the eigenvalues of A don't have to be real. Sometimes, we want to find a "simple" form for A that does have real entries. What follows is a way to tweak the Jordan canonical form to this use case.

We observe the following:

Lemma 1.3.1. Let $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{C}$. Then, the "Jordan structure of A at λ" (meaning the multiset of the sizes of the Jordan blocks of A at λ) equals the Jordan structure of A at $\overline{\lambda}$. In other words, for each $p > 0$, we have

$$\begin{aligned} &(\text{the number of Jordan blocks of } A \text{ at } \lambda \text{ having size } p) \\ &= \left(\text{the number of Jordan blocks of } A \text{ at } \overline{\lambda} \text{ having size } p\right). \end{aligned}$$

In other words, Jordan blocks at λ and Jordan blocks at $\overline{\lambda}$ come in pairs of equal sizes.

Proof. HW. $\square$

So we can try to combine each Jordan block at λ with an equally sized Jordan block at $\overline{\lambda}$ and hope that something real comes out somehow, in the same way as $(t-\lambda)\left(t-\overline{\lambda}\right) = t^2 - 2\operatorname{Re}\lambda + |\lambda|^2 \in \mathbb{R}[t]$.

How to do this?

$$
\begin{pmatrix}
\lambda & 1 & & & & & & \\
 & \lambda & \ddots & & & & & \\
 & & \ddots & 1 & & & & \\
 & & & \lambda & & & & \\
 & & & & \overline{\lambda} & 1 & & \\
 & & & & & \overline{\lambda} & \ddots & \\
 & & & & & & \ddots & 1 \\
 & & & & & & & \overline{\lambda}
\end{pmatrix}
$$

$$
\sim \begin{pmatrix}
\lambda & & 1 & & & & & \\
 & \overline{\lambda} & & 1 & & & & \\
 & & \lambda & & 1 & & & \\
 & & & \overline{\lambda} & & 1 & & \\
 & & & & \ddots & & & \\
 & & & & & \ddots & & \\
 & & & & & & \lambda & \\
 & & & & & & & \overline{\lambda}
\end{pmatrix}
= \begin{pmatrix}
L & I_2 & & \\
 & L & I_2 & \\
 & & \ddots & \\
 & & & L
\end{pmatrix},
$$

where L is the 2×2-matrix $\begin{pmatrix} \lambda & \\ & \overline{\lambda} \end{pmatrix}$. However,

$$
\begin{pmatrix} \lambda & \\ & \overline{\lambda} \end{pmatrix} \sim \begin{pmatrix} a & b \\ -b & a \end{pmatrix},
$$

where $a = \operatorname{Re}\lambda$ and $b = \operatorname{Im}\lambda$ (so that $\lambda = a + bi$). (HW!) So our matrix is similar to

$$
\begin{pmatrix}
a & b & 1 & & & & & \\
-b & a & & 1 & & & & \\
 & & a & b & 1 & & & \\
 & & -b & a & & 1 & & \\
 & & & & \ddots & & & \\
 & & & & & \ddots & & \\
 & & & & & & a & b \\
 & & & & & & -b & a
\end{pmatrix}.
$$