Math 504: Advanced Linear Algebra

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Math 504 Lecture 5

1. Schur triangularization (cont'd)

1.1. Normal matrices

Definition 1.1.1. A square matrix $A \in \mathbb{C}^{n \times n}$ is said to be **normal** if $AA^* = A^*A$.

In other words, a square matrix is normal if it commutes with its own conjugate transpose. Here are some examples:

Example 1.1.2. (a) Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Then, $A^* = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $AA^* = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $A^*A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, so that $AA^* = A^*A$. Thus, A is normal.

(b) Let $B = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$. Then, $B^* = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}$ and $BB^* = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B^*B = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, so that $BB^* \neq B^*B$. Thus, B is not normal.

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(c) For any $a, b \in \mathbb{C}$, the matrix $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ is normal.

The class of normal matrices includes several known classes. Recall:

- A square matrix $A \in \mathbb{C}^{n \times n}$ is unitary if and only if $AA^* = A^*A = I_n$.
- A square matrix $A \in \mathbb{C}^{n \times n}$ is **Hermitian** if and only if $A^* = A$.
- A square matrix $A \in \mathbb{C}^{n \times n}$ is **skew-Hermitian** if and only if $A^* = -A$.

Proposition 1.1.3. **(a)** Every Hermitian matrix is normal.
(b) Every skew-Hermitian matrix is normal.
(c) Every unitary matrix is normal.
(d) Every diagonal matrix is normal.

Proof. **(a)** If A is Hermitian, then $A^* = A$, so that

$$A \underbrace{A^*}_{=A} = \underbrace{A}_{=A^*} A = A^* A,$$

so that A is normal.
(b) Similar.
(c) Trivial.
(d) LTTR. □

Unlike the unitary matrices, the normal matrices are not closed under multiplication (i.e., A and B can be unitary without AB being unitary).

Here are two more ways to construct normal matrices out of existing normal matrices:

Proposition 1.1.4. Let $A \in \mathbb{C}^{n \times n}$ be a normal matrix.
(a) If $\lambda \in \mathbb{C}$ is arbitrary, then $\lambda I_n + A$ is normal.
(b) If $U \in \mathbb{C}^{n \times n}$ is a unitary matrix, then the matrix UAU^* is normal.

Proof. We have $AA^* = A^*A$ (since A is normal).
(a) Let $\lambda \in \mathbb{C}$. Then,

$$(\lambda I_n + A)^* = (\lambda I_n)^* + A^* = \overline{\lambda} I_n + A^*.$$

Hence,

$$(\lambda I_n + A)(\lambda I_n + A)^* = (\lambda I_n + A)\left(\overline{\lambda} I_n + A^*\right) = \lambda \overline{\lambda} I_n + \lambda A^* + \overline{\lambda} A + AA^*$$

and similarly

$$(\lambda I_n + A)^* (\lambda I_n + A) = \overline{\lambda} \lambda I_n + \lambda A^* + \overline{\lambda} A + A^* A.$$

The right hand sides are equal, since $\lambda\overline{\lambda} = \overline{\lambda}\lambda$ and $AA^* = A^*A$. Thus, the left hand sides are equal, too. In other words,

$$(\lambda I_n + A)(\lambda I_n + A)^* = (\lambda I_n + A)^*(\lambda I_n + A).$$

So $\lambda I_n + A$ is normal.

(b) Let U be a unitary matrix. Then, $U^*U = UU^* = I_n$. Now,

$$(UAU^*)^* = \underbrace{(U^*)^*}_{=U} A^*U^* = UA^*U^*.$$

Thus,

$$(UAU^*)(UAU^*)^* = (UAU^*)(UA^*U^*) = UA\underbrace{U^*U}_{=I_n}A^*U^* = UAA^*U^*.$$

Similarly,

$$(UAU^*)^*(UAU^*) = UA^*AU^*.$$

Again, the right hand sides are equal, since $AA^* = A^*A$. So the left hand sides are equal, and this shows that UAU^* is normal. □

We will now show the following:

Lemma 1.1.5. Let $T \in \mathbb{C}^{n\times n}$ be a triangular matrix. Then, T is normal if and only if T is diagonal.

Proof. $\Longleftarrow$: If T is diagonal, then T is normal, as we have already seen.

$\Longrightarrow$: Assume that T is normal. We must show that T is diagonal.

WLOG assume that T is upper-triangular (since the other case is analogous).

Write T in the form

$$T = \begin{pmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,n} \\ & t_{2,2} & \cdots & t_{2,n} \\ & & \ddots & \vdots \\ & & & t_{n,n} \end{pmatrix},$$

where the invisible entries are 0's. (We can do this, since T is upper-triangular.) Thus,

$$T^* = \begin{pmatrix} \overline{t_{1,1}} & & & \\ \overline{t_{1,2}} & \overline{t_{2,2}} & & \\ \vdots & \vdots & \ddots & \\ \overline{t_{1,n}} & \overline{t_{2,n}} & \cdots & \overline{t_{n,n}} \end{pmatrix}.$$

Since T is normal, we have $TT^* = T^*T$. Now let's look at the entries of this matrix. We have

$$(TT^*)_{1,1} = t_{1,1}\overline{t_{1,1}} + t_{1,2}\overline{t_{1,2}} + \cdots + t_{1,n}\overline{t_{1,n}} = |t_{1,1}|^2 + |t_{1,2}|^2 + \cdots + |t_{1,n}|^2, \qquad \text{but}$$
$$(T^*T)_{1,1} = \overline{t_{1,1}}t_{1,1} = |t_{1,1}|^2.$$

However, the left hand sides of these must be equal, since $TT^* = T^*T$. Thus, the right hand sides are equal too. That is,

$$|t_{1,1}|^2 + |t_{1,2}|^2 + \cdots + |t_{1,n}|^2 = |t_{1,1}|^2 .$$

Thus, $|t_{1,2}|^2 + \cdots + |t_{1,n}|^2 = 0$, so that $t_{1,2} = t_{1,3} = \cdots = t_{1,n} = 0$ (because $|t_{1,i}|^2 \geq 0$ for all i).

We continue with the $2,2$-entries:

$$(TT^*)_{2,2} = t_{2,2}\overline{t_{2,2}} + t_{2,3}\overline{t_{2,3}} + \cdots + t_{2,n}\overline{t_{2,n}} = |t_{2,2}|^2 + |t_{2,3}|^2 + \cdots + |t_{2,n}|^2 \qquad \text{and}$$
$$(T^*T)_{2,2} = \overline{t_{1,2}}t_{1,2} + \overline{t_{2,2}}t_{2,2} = \underbrace{|t_{1,2}|^2}_{\substack{=0 \\ (\text{since } t_{1,2}=0)}} + |t_{2,2}|^2 = |t_{2,2}|^2 .$$

Comparing these equalities, we get

$$|t_{2,2}|^2 + |t_{2,3}|^2 + \cdots + |t_{2,n}|^2 = |t_{2,2}|^2 .$$

As before, this lets us conclude that $t_{2,3} = t_{2,4} = \cdots = t_{2,n} = 0$.

Keep going like this to prove that

$$t_{i,j} = 0 \qquad \text{for all } i < j.$$

Strictly speaking, this is a strong induction on i. In the induction step, use the fact that $t_{k,i} = 0$ for all $k < i$ (this follows from the induction hypothesis).

This shows that T is a diagonal matrix, qed. □

1.2. The spectral theorem

The spectral theorem provides an answer to the question "what are normal matrices **really**?".

Theorem 1.2.1 (spectral theorem for normal matrices)**.** Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix. Then:

(a) There exists a unitary matrix $U \in \mathrm{U}_n(\mathbb{C})$ and a diagonal matrix $D \in \mathbb{C}^{n\times n}$ such that

$$A = UDU^*.$$

In other words, A is unitarily similar to a diagonal matrix.

(b) Let $U \in \mathrm{U}_n(\mathbb{C})$ be a unitary matrix and $D \in \mathbb{C}^{n\times n}$ be a diagonal matrix such that $A = UDU^*$. Then, the diagonal entries of D are the eigenvalues of A. Moreover, the columns of U are eigenvectors of A. Thus, there exists an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A.

Proof. **(a)** The Schur triangularization theorem tells us that we can write A in the form

$$A = UTU^*$$

for a unitary matrix U and an upper-triangular matrix T. Consider these U and T. If we can show that T is diagonal, then we are done.

From $A = UTU^*$, we see that A is unitarily equivalent to T. Thus, T is unitarily equivalent to A. Hence, our proposition above yields that T is normal, since A is normal. Now, the Lemma we just proved yields that T is diagonal (because T is triangular and normal). So part **(a)** is proven.

(b) The matrix D is unitarily similar to A (since $A = UDU^*$), thus similar to A. Hence, D has the same eigenvalues as A. However, D is a diagonal matrix, so its eigenvalues are its diagonal entries. So the diagonal entries of D must be the eigenvalues of A.

What about U ? The columns of U are $Ue_1, Ue_2, \ldots, Ue_n$, where $(e_1, e_2, \ldots, e_n)$ is the standard basis of $\mathbb{C}^n$.

[In general, for example, $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} e_2 = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ e \\ h \end{pmatrix}$.]

Now, I claim that each Ue_i is an eigenvector of A. Indeed,

$$\underbrace{A}_{=UDU^*} \cdot Ue_i = UD \underbrace{U^*U}_{=I_n} e_i = U \underbrace{De_i}_{\substack{=\lambda e_i \\ \text{where } \lambda \text{ is the } (i,i)\text{-th entry of } D \\ \text{(since } D \text{ is diagonal)}}} = U \cdot \lambda e_i = \lambda \cdot Ue_i.$$

Thus, we conclude that the n columns of U are eigenvectors of A.

Since U is unitary, these n columns form an orthonormal basis of $\mathbb{C}^n$. Thus, we have found an orthonormal basis of $\mathbb{C}^n$ that consists of eigenvectors of A. Qed. □

The decomposition $A = UDU^*$ in the spectral theorem (or, to be more precise, the pair (U, D)) is called a **spectral decomposition** of A.

Only normal matrices have a spectral decomposition:

Corollary 1.2.2. An $n \times n$-matrix $A \in \mathbb{C}^{n \times n}$ is normal if and only if it is unitarily similar to a diagonal matrix.

Proof. $\Longrightarrow$: This is just part **(a)** of the spectral theorem.

$\Longleftarrow$: Assume that A is unitarily similar to a diagonal matrix. Thus, A is unitarily similar to a normal matrix (since diagonal matrices are normal), and thus itself normal. Qed. □

We can similarly characterize Hermitian matrices:

Proposition 1.2.3. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix, and let (U, D) be a spectral decomposition of A. Then, the diagonal entries of D are real.

Proof. We have $A = UDU^*$, thus $A^* = (UDU^*)^* = UD^*U^*$. However, since A is Hermitian, we have $A^* = A$. In other words, $UD^*U^* = UDU^*$. We can cancel U and U^* from this equality (since U is unitary), and get $D^* = D$. Since D is diagonal, this is simply saying that each diagonal entry of D equals its own conjugate, i.e., is real. □

Corollary 1.2.4. An $n \times n$-matrix $A \in \mathbb{C}^{n \times n}$ is Hermitian if and only if it is unitarily similar to a diagonal matrix with real entries.

Proof. $\Longrightarrow$: Follows from the preceding proposition + the spectral theorem.

$\Longleftarrow$: If A is unitarily similar to a diagonal matrix with real entries, then $A = UDU^*$ where D is a diagonal matrix with real entries. Thus, $A^* = UD^*U^*$. However, $D^* = D$. So it follows that $A^* = A$, so A is Hermitian. $\square$

The corollary we just proved is the complex analogue of the classical "real spectral theorem", which says that a symmetric matrix $A \in \mathbb{R}^{n \times n}$ is similar to a diagonal matrix with real entries via an orthogonal matrix with real entries.

Similarly, we can handle skew-Hermitian matrices:

Proposition 1.2.5. Let $A \in \mathbb{C}^{n \times n}$ be a skew-Hermitian matrix, and let (U, D) be a spectral decomposition of A. Then, the diagonal entries of D are purely imaginary.

Corollary 1.2.6. An $n \times n$-matrix $A \in \mathbb{C}^{n \times n}$ is skew-Hermitian if and only if it is unitarily similar to a diagonal matrix with purely imaginary entries.

Likewise, we can handle unitary matrices:

Proposition 1.2.7. Let $A \in \mathbb{C}^{n \times n}$ be a unitary matrix, and let (U, D) be a spectral decomposition of A. Then, the diagonal entries of D are complex numbers with absolute value 1.

Corollary 1.2.8. An $n \times n$-matrix $A \in \mathbb{C}^{n \times n}$ is unitary if and only if it is unitarily similar to a diagonal matrix whose diagonal entries have absolute value 1.