10. Math 235 Fall 2021, Worksheet A: assorted problems

This is a selection of contest-type problems for discussion.

Exercise 10.0.1. (Stronger version of Putnam 2002 Problem A5)
Consider the sequence $(u_0, u_1, u_2, \ldots)$ of numbers defined recursively by

$$\begin{aligned} u_0 &= 1; \\ u_{2n} &= u_n + u_{n-1} \qquad \text{for all positive } n \in \mathbb{N}; \\ u_{2n+1} &= u_n \qquad \text{for all } n \in \mathbb{N}. \end{aligned}$$

(As we recall, $\mathbb{N} = \{0, 1, 2, \ldots\}$.) Prove that for each positive rational number k, there is a unique $n \in \mathbb{N}$ such that $u_n / u_{n+1} = k$.

Exercise 10.0.2. **(a)** For which nonnegative integers n is the polynomial $x^n + 1$ a multiple of the polynomial $x + 1$?

(b) For which nonnegative integers n is the polynomial $x^{2n} + x^n + 1$ a multiple of the polynomial $x^2 + x + 1$?

(c) Fix a positive integer m. For which nonnegative integers n is the polynomial $x^{(m-1)n} + x^{(m-2)n} + \cdots + x^n + 1$ a multiple of the polynomial $x^{m-1} + x^{m-2} + \cdots + x + 1$?

Exercise 10.0.3. Let $n \in \mathbb{N}$. Assume that the set $\{1, 2, \ldots, 2n\}$ is partitioned into two disjoint subsets A and B, each of which has size n.
Let $a_1, a_2, \ldots, a_n$ be the elements of A in increasing order (that is, $a_1 < a_2 < \cdots < a_n$).
Let $b_1, b_2, \ldots, b_n$ be the elements of B in decreasing order (that is, $b_1 > b_2 > \cdots > b_n$).
Prove that

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n| = n^2.$$

Exercise 10.0.4. Let a, b, c, d be four integers such that $ab = cd$. Prove that there exist four integers x, y, z, w such that

$$a = xy, \qquad b = zw, \qquad c = xz, \qquad d = yw.$$

Exercise 10.0.5. Fix an $n \in \mathbb{N}$. Let $[n]$ be the set $\{1, 2, \ldots, n\}$.

The *top-led shuffle* is the operation that takes an n-tuple $(a_1, a_2, \ldots, a_n)$ of elements of $[n]$ and changes it by moving its first entry a_1 to the a_i-th position (i.e., it replaces the n-tuple by $(a_2, a_3, \ldots, a_k, a_1, a_{k+1}, a_{k+2}, \ldots, a_n)$, where $k = a_1$). In other words, if we view the n-tuple $(a_1, a_2, \ldots, a_n)$ as a deck of n cards (numbered $a_1, a_2, \ldots, a_n$ from top to bottom), then the top-led shuffle moves the topmost card to the position indicated by the number on the card (so, e.g., if the number on the card is 3, then it moves this card to the 3-rd position).

[**Example:** The top-led shuffle turns $(4, 1, 5, 2, 3)$ into $(1, 5, 2, 4, 3)$.]

Start with some n-tuple $(a_1, a_2, \ldots, a_n)$ of elements of $[n]$ that contains each $i \in [n]$ exactly once (i.e., start with a permutation of the list $(1, 2, \ldots, n)$), and apply the top-led shuffle to it over and over.

Prove that eventually, this n-tuple will stabilize (i.e., will no longer change under top-led shuffles).

[**Remark:** In contrast, if we start with an arbitrary n-tuple $(a_1, a_2, \ldots, a_n)$ of elements of $[n]$, then the top-led shuffle might run in circles rather than stabilizing. For an example, try $n = 4$ and $(a_1, a_2, \ldots, a_n) = (2, 3, 3, 1)$.]

Exercise 10.0.6. Let $n \in \mathbb{N}$. Prove that $\left\lfloor 2 + \sqrt{3} \right\rfloor^n$ is odd. (Here, for any given real x, the notation $\lfloor x \rfloor$ stands for the largest integer that is $\leq x$.)