Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2021: Homework 6

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October 30, 2021

1 Exercise 1

1.1 Problem

Consider a conference with an even (and nonzero) number of participants. Prove that there exist two distinct participants that have an even number of common friends.

(We are assuming here that friendship is a mutual relation – i.e., if a is a friend of b, then b is a friend of a. Furthermore, no person a counts as his own friend.)

1.2 Solution

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2 Exercise 2

2.1 Problem

Let $G = (V, E)$ be a simple graph. Set $n = |V|$. Prove that we can find some edges $e_1, e_2, \ldots, e_k$ of G and some triangles $t_1, t_2, \ldots, t_\ell$ of G such that $k + \ell \leq n^2/4$ and such that each edge $e \in E \setminus \{e_1, e_2, \ldots, e_k\}$ is a subset of (at least) one of the triangles $t_1, t_2, \ldots, t_\ell$.

2.2 Solution

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3 Exercise 3

3.1 Problem

Let p and q be two positive integers. Let $g = \gcd(p, q)$. Let $n \in \mathbb{N}$ satisfy $n \geq p + q - g$. Let $(x_1, x_2, \ldots, x_n)$ be an n-tuple of arbitrary objects. Assume that

$$x_i = x_{i+p} \qquad \text{for each } i \in [n-p].$$

Assume furthermore that

$$x_i = x_{i+q} \qquad \text{for each } i \in [n-q].$$

Then, prove that $x_i = x_{i+g}$ for each $i \in [n-g]$.

3.2 Solution

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4 Exercise 4

4.1 Problem

Let V be a nonempty finite set. Let G and H be two graphs with vertex set V. Assume that for each $u \in V$ and $v \in V$, there exists a path from u to v in G or a path from u to v in H. Prove that at least one of the graphs G and H is connected.

4.2 Solution

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5 Exercise 5

5.1 Problem

Let G be a graph. Let $d > 2$ be an integer. Assume that each vertex of G has degree $\geq d$. Prove that G has a cycle whose length is not divisible by d.

5.2 Solution

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6 Exercise 6

6.1 Problem

Among n senators, some are enemies. It is assumed that the "enemy' relation is mutual – i.e., if a is an enemy of b, then b is an enemy of a. A set S of senators is said to be *odious* if each senator not in S has at least one enemy in S. Prove that the number of odious sets of senators is odd.

(Note that the set of all n senators is always odious, for vacuous reasons.)

[**Example:** If $n = 3$ and the three senators are labelled $1, 2, 3$, and the only pairs of mutual enemies are $\{1, 2\}$ and $\{1, 3\}$, then the odious sets are $\{1\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$.]

6.2 Solution

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References

[Grinbe20] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 10 August 2021. `https://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`