Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2021: Homework 5

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1 Exercise 1

1.1 Problem

Let $a, b, c \in \mathbb{Z}$. Prove that:

(a) We have

$$\gcd(b,c) \cdot \gcd(c,a) \cdot \gcd(a,b) = \gcd(a,b,c) \cdot \gcd(bc,ca,ab).$$

(b) We have

$$\operatorname{lcm}(b,c) \cdot \operatorname{lcm}(c,a) \cdot \operatorname{lcm}(a,b) = \operatorname{lcm}(a,b,c) \cdot \operatorname{lcm}(bc,ca,ab).$$

(c) Assume that a, b, c are nonzero. Then,

$$\frac{\gcd(bc,ca,ab)}{\gcd(a,b,c)} = \frac{\operatorname{lcm}(bc,ca,ab)}{\operatorname{lcm}(a,b,c)} \in \mathbb{Z}.$$

1.2 Solution

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2 Exercise 2

2.1 Problem

Fix a prime p. Let a and b be two integers satisfying $\gcd(a, b) = p$. What values can $\gcd(a^5, b^{13})$ take?

2.2 Solution

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3 Exercise 3

3.1 Problem

Let $n, a, b, c \in \mathbb{N}$ such that n is odd. Prove that

$$\frac{(na)!\,(nb)!\,(nc)!}{a!b!c!\,((b+c)!\,(c+a)!\,(a+b)!)^{(n-1)/2}}$$

is an integer.

3.2 Solution

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$(n+1)\operatorname{lcm}\left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\right) = \operatorname{lcm}(1, 2, \ldots, n+1).$$

4.2 Solution

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5 Exercise 5

5.1 Problem

Let k and n be any two positive integers. Prove that an expression of the form

$$\pm\frac{1}{k} \pm \frac{1}{k+1} \pm \frac{1}{k+2} \pm \cdots \pm \frac{1}{k+n}$$

(where each $\pm$ sign is either a $+$ or a $-$ sign) will never be an integer, no matter what the $\pm$ signs are.

[**Hint:** Show that one of the numbers $k, k+1, \ldots, k+n$ has a higher 2-valuation than all of the others.]

5.2 Solution

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6 Exercise 6

6.1 Problem

Let $(u_1, u_2, u_3, \ldots)$ be a sequence of nonzero integers such that

$$\text{every } a, b \in \{1, 2, 3, \ldots\} \text{ satisfy } \gcd(u_a, u_b) = \left|u_{\gcd(a,b)}\right|.$$

(We have seen such a sequence in Exercise 3.7.4; another example is the Fibonacci sequence because of [Grinbe20, Exercise 3.7.2].)

Prove that there exists a sequence $(v_1, v_2, v_3, \ldots)$ of nonzero integers such that

$$\text{each } n \in \{1, 2, 3, \ldots\} \text{ satisfies } u_n = \prod_{d \mid n} v_d.$$

Here, the symbol "$\prod_{d \mid n}$" means a product over all positive divisors d of n. Thus, the equality $u_n = \prod_{d \mid n} v_d$ means

$$\begin{aligned}
u_1 &= v_1 && \text{for } n = 1; \\
u_2 &= v_1 v_2 && \text{for } n = 2; \\
u_3 &= v_1 v_3 && \text{for } n = 3; \\
u_4 &= v_1 v_2 v_4 && \text{for } n = 4; \\
u_5 &= v_1 v_5 && \text{for } n = 5; \\
u_6 &= v_1 v_2 v_3 v_6 && \text{for } n = 6;
\end{aligned}$$

and so on.

6.2 Solution

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References

[Grinbe20] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 10 August 2021. `https://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`