Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2021: Homework 2

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1 Exercise 1

1.1 Problem

Let S be a 10-element subset of the set $\{1, 2, \ldots, 100\}$. Prove that there exist two disjoint nonempty subsets A and B of S such that $\sum_{a \in A} a = \sum_{b \in B} b$. (Note that A and B are allowed to have any positive sizes, including 1.)

[**Example:** If $S = \{3, 9, 13, 19, 26, 60, 74, 80, 84, 94\}$, then $\sum_{a \in A} a = \sum_{b \in B} b$ is satisfied for $A = \{3, 9, 94\}$ and $B = \{26, 80\}$ (as well as for various other choices).]

1.2 Solution

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2 Exercise 2

2.1 Problem

A number of people have been settled in n apartments $B_1, B_2, \ldots, B_n$ (with each person settled in exactly one apartment). (Roommates are allowed.) Now, all these people are removed from their apartments and resettled in $n + 1$ new apartments $C_1, C_2, \ldots, C_{n+1}$ in such a way that none of these $n + 1$ new apartments stays empty.

A person is said to have *gained space* if he has fewer roommates after the resettlement than he used to have before.

Prove that at least two people have gained space.

2.2 Solution

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3 Exercise 3

3.1 Problem

Consider any six points on the circumference of a circle with radius 1. Prove that some two of these six points have distance ≤ 1.

3.2 Solution

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4 Exercise 4

4.1 Problem

Let N be an n-element set with $n > 0$. Let $A_1, A_2, \ldots, A_q$ be finitely many 2-element subsets of N. Let $m = \left\lceil \frac{2q}{n} \right\rceil$. Prove that we can find a strictly increasing sequence $(i_1 < i_2 < \cdots < i_m)$ of m elements of $\{1, 2, \ldots, q\}$ such that

$$\left|A_{i_j} \cap A_{i_{j+1}}\right| \geq 1 \qquad \text{for each } j \in \{1, 2, \ldots, m-1\}.$$

[**Example:** Let $n = 5$ and $N = \{1, 2, 3, 4, 5\}$ and $q = 6$ and

$$A_1 = \{1, 2\}, \qquad A_2 = \{3, 5\}, \qquad A_3 = \{1, 4\},$$
$$A_4 = \{2, 3\}, \qquad A_5 = \{3, 5\}, \qquad A_6 = \{4, 5\}.$$

We have $m = \left\lceil \frac{2q}{n} \right\rceil = \left\lceil \frac{2 \cdot 6}{5} \right\rceil = 3$. Thus, the exercise claims that there exists a strictly increasing sequence $(i_1 < i_2 < i_3)$ of 3 elements of $\{1, 2, 3, 4, 5, 6\}$ such that $|A_{i_1} \cap A_{i_2}| \geq 1$ and $|A_{i_2} \cap A_{i_3}| \geq 1$. And indeed, we can pick $i_1 = 1$ and $i_2 = 3$ and $i_3 = 6$ for example.]

4.2 Solution

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5 Exercise 5

5.1 Problem

Let A be a rectangular matrix with real entries. Assume that the entries in each **row** of A are weakly increasing from left to right. Now we sort the entries in each **column** of A so that they become weakly increasing from top to bottom. (Each entry stays within its column.) Thus, we obtain a new matrix B. Prove that the entries in each **row** of B are still weakly increasing from left to right.

[**Example:** If $A = \begin{pmatrix} 1 & 3 & 6 \\ 2 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix}$, then $B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 3 & 6 \end{pmatrix}$.]

5.2 Solution

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6 Exercise 6

6.1 Problem

Let $\mathbf{q} = (q_1, q_2, q_3, \ldots)$ be a sequence of positive integers that is weakly increasing (that is, we have $q_1 \leq q_2 \leq q_3 \leq \cdots$) and unbounded from above (that is, for each positive integer N, there exists some $i \geq 1$ such that $q_i \geq N$).

For each positive integer k, let $t(k)$ be the number of entries of $\mathbf{q}$ that do not exceed k (that is, the number of all positive integers i such that $q_i \leq k$).

Prove that the two sets

$$Q := \{q_n + n - 1 \mid n \text{ is a positive integer}\} = \{q_1 + 0, \quad q_2 + 1, \quad q_3 + 2, \quad \ldots\}$$

and

$$T := \{t(n) + n \mid n \text{ is a positive integer}\} = \{t(1) + 1, \ t(2) + 2, \ t(3) + 3, \ \ldots\}$$

are disjoint and their union is $\{1, 2, 3, \ldots\}$.

[**Example:** If $q_i = i$ for each i, then $t(k) = k$ for each k, and we have

$$Q = \{\text{odd positive integers}\} \qquad \text{and} \qquad T = \{\text{even positive integers}\}.$$

If $q_i = pi$ for each i for some fixed positive integer p, then $t(k) = \left\lfloor \dfrac{k}{p} \right\rfloor$ for each k, and we have

$$Q = \{\text{positive integers that leave the remainder } p \text{ when divided by } p+1\}$$

and

$$T = \{\text{all other positive integers}\},$$

although this takes some moments to convince yourself of.

If $q_i = i^2$ for each i, then $t(k) = \left\lfloor \sqrt{k} \right\rfloor$ for each k, and we have

$$Q = \left\{ n^2 + n - 1 \mid n \text{ is a positive integer} \right\} = \{1, 5, 11, 19, \ldots\}$$

and

$$T = \left\{ \left\lfloor \sqrt{n} \right\rfloor + n \mid n \text{ is a positive integer} \right\} = \{2, 3, 4, 6, 7, \ldots\}.$$

Many other examples can be constructed (and have, in fact, appeared on contests!).]

6.2 Solution

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References