Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2020: Homework 9

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1 Exercise 1

1.1 Problem

Let n and d be two positive integers.

An n-tuple $(x_1, x_2, \ldots, x_n) \in [d]^n$ will be called *first-even* if its first entry x_1 occurs in it an even number of times (i.e., the number of $i \in [n]$ satisfying $x_i = x_1$ is even). (For example, the 3-tuples $(1, 5, 1)$ and $(2, 2, 3)$ are first-even, while the 3-tuple $(4, 1, 1)$ is not.)

Compute the # of first-even n-tuples in $[d]^n$.

1.2 Solution

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$ and $p \in \mathbb{Z}$. Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k}{n+p} = 2^{n-p} \binom{n}{p}.$$

2.2 Solution

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3 Exercise 3

3.1 Problem

Let n be a positive integer. Prove that

$$\sum_{i=0}^{n-1} \frac{(-1)^i}{n-i} \binom{n-i}{i} = \frac{1}{n} (-1)^{(n+1)//3} (1 + [3 \mid n]) .$$

3.2 Solution

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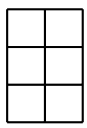
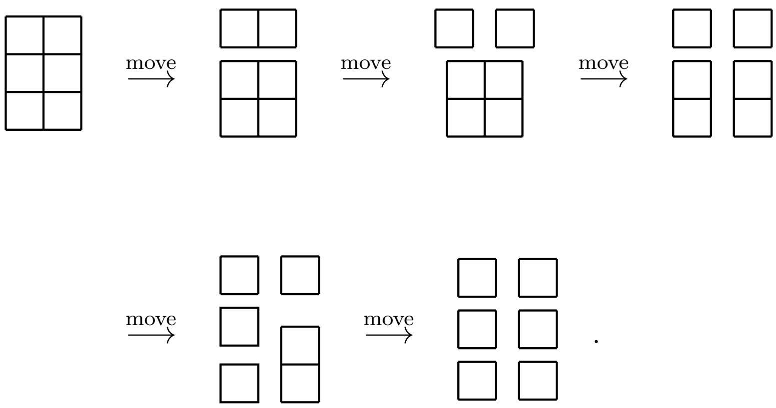
4 Exercise 4

4.1 Problem

A chocolate bar has the shape of an $m \times n$-rectangle (subdivided into little 1×1-squares by horizontal and vertical lines, in case your eating habits are too healthy). For example, if $m = 2$ and $n = 3$, then it looks like this:



In one move, you can break a chocolate bar into two by splitting it along one of the (horizontal or vertical) lines that divide it (unless it is already a 1×1-square). For example:



(a) What is the smallest number of moves necessary to break up the entire bar into 1×1-squares?

(b) What is the largest number of moves necessary to break up the entire bar into 1×1-squares?

4.2 Solution

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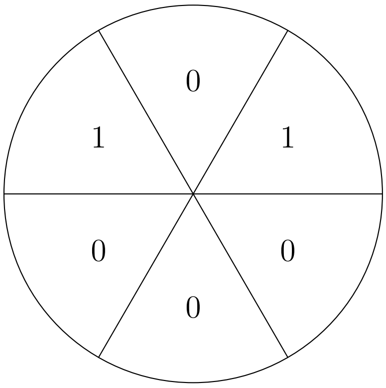
5 Exercise 5

5.1 Problem

A circle is split into 6 sectors, and a number has been written in each sector. These numbers are $1, 0, 1, 0, 0, 0$ in clockwise order, as shown in the following picture:



In one move, you can add 1 to any two numbers written in adjacent sectors.

Can you ever ensure that all six sectors have the same number written in them?

5.2 Solution

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6 Exercise 6

6.1 Problem

Fix a positive integer n. Consider n chips placed in a heap. In a move, you are allowed to split a heap H of chips into two smaller heaps H_1 and H_2; when doing so, you gain $|H_1| \cdot |H_2|$ cents. (We treat heaps as sets of chips; thus, $|H_1|$ is the # of chips in heap H_1.) After sufficiently many moves, you are left with n heaps, each containing exactly one chip. What is the maximum number of cents you can have made by that moment?

6.2 Solution

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7 Exercise 7

7.1 Problem

Fix two positive integers a and b. Consider $a+b$ bowls, numbered $1, 2, \ldots, a+b$. Initially, each of the bowls $1, 2, \ldots, a$ contains an apple, and each of the bowls $a+1, a+2, \ldots, a+b$ contains a pear. A *move* consists of picking two numbers $i, j \in [a+b]$ satisfying $i < a+b$ and $j > 1$ and $i \equiv j \mod 2$, and moving an apple from bowl i to bowl $i+1$ and a pear from bowl j to bowl $j-1$. (We assume that these fruits do exist in these bowls; otherwise, the move cannot be made. It is allowed for several fruits to lie in one bowl at the same time.)

The goal is to end up with each of the bowls $1, 2, \ldots, b$ containing a pear and each of the bowls $b+1, b+2, \ldots, b+a$ containing an apple. Show that this goal can be reached if and only if the product ab is even.

7.2 Solution

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8 Exercise 8

8.1 Problem

Consider a hotel with an infinite number of rooms, arranged sequentially on the ground floor. The rooms are labelled by integers $i \in \mathbb{Z}$, with room i being adjacent to rooms $i-1$ and $i+1$. A finite number of violinists are staying in the hotel (it is possible for two violinists to be staying in the same room). Each night, two violinists staying in the same room decide they cannot stand each other's noise, and move to the two adjacent rooms (i.e., if they were in room i, they move to rooms $i-1$ and $i+1$). This keeps happening for as long as there are two violinists staying in the same room.

Prove that this moving will stop after a finite number of days (i.e., there will be a day when no two violinists share a room any more).

8.2 Solution

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9 Exercise 9

9.1 Problem

Define a sequence $(t_0, t_1, t_2, \ldots)$ of positive rational numbers recursively by setting

$$t_n = 1 \qquad \text{for each } n < 4$$

and

$$t_n = \frac{1 + t_{n-1}t_{n-3}}{t_{n-4}} \qquad \text{for each } n \geq 4.$$

Prove that t_n is a positive integer for each integer $n \geq 0$.

9.2 Solution

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10 Exercise 10

10.1 Problem

Let $q \in \mathbb{N}$. Define a sequence $(t_0, t_1, t_2, \ldots)$ of positive rational numbers recursively by setting

$$t_n = 1 \qquad \text{for each } n < 3$$

and

$$t_n = \frac{t_{n-1}^2 + q t_{n-1} t_{n-2} + t_{n-2}^2}{t_{n-3}} \qquad \text{for each } n \geq 3.$$

Prove that t_n is a positive integer for each integer $n \geq 0$.

10.2 Solution

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References