Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2020: Homework 7

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1 Exercise 1

1.1 Problem

Let U and V be two finite sets. Let $f : U \to V$ and $g : V \to U$ be two maps. Prove that $|\operatorname{Fix}(f \circ g)| = |\operatorname{Fix}(g \circ f)|$.

1.2 Solution

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2 Exercise 2

2.1 Problem

Let n be a positive integer. Prove that the number of **even** positive divisors of n is even if and only if $n/2$ is not a perfect square.

2.2 Solution

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3 Exercise 3

3.1 Problem

Let n and r be positive integers. Prove that

$$\sum_{\substack{S \subseteq [n]; \\ |S| = r}} \min S = \binom{n+1}{r+1}.$$

(Recall that the summation sign "$\sum\limits_{\substack{S \subseteq [n]; \\ |S| = r}}$" means "sum over all subsets S of $[n]$ satisfying $|S| = r$", that is, "sum over all r-element subsets S of $[n]$".)

3.2 Solution

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4 Exercise 4

4.1 Problem

Let n be a positive integer. Find the # of compositions of n that don't contain 1 as an entry.

(For example, if $n = 7$, then the compositions of n that don't contain 1 as an entry are (7), $(5, 2)$, $(4, 3)$, $(3, 4)$, $(2, 5)$, $(3, 2, 2)$, $(2, 3, 2)$ and $(2, 2, 3)$.)

4.2 Solution

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5 Exercise 5

5.1 Problem

Let n be a positive integer. Let $k \in \mathbb{N}$.

(a) A *composition of n into k parts* means a composition of n that has exactly k entries (i.e., is a k-tuple). Find the # of compositions of n into k parts.

(b) A *weak composition of n into k parts* means a k-tuple of **nonnegative** integers whose sum is n. (So it is like a composition of n into k parts, but its entries are allowed to be 0.) Find the # of weak compositions of n into k parts.

5.2 Solution

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6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(k+1)!}{n^k} = n.$$

6.2 Solution

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7 Exercise 7

7.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} k \binom{2n}{n-k} = n \binom{2n-1}{n}.$$

7.2 Solution

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8 Exercise 8

8.1 Problem

Let $(f_0, f_1, f_2, \ldots)$ and $(g_0, g_1, g_2, \ldots)$ be two sequences of numbers. Assume that

$$g_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f_i \qquad \text{for every } n \in \mathbb{N}.$$

Prove that

$$f_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} g_i \qquad \text{for every } n \in \mathbb{N}.$$

8.2 Solution

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9 Exercise 9

9.1 Problem

Let $n \in \mathbb{N}$.

(a) Prove that

$$\binom{-1/2}{n} = \left(\frac{-1}{4}\right)^n \binom{2n}{n}.$$

(b) Prove that

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k} = 4^n.$$

9.2 Solution

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10 Exercise 10

10.1 Problem

Prove that there is a **unique** sequence $(u_0, u_1, u_2, \ldots)$ of positive integers such that

$$u_n^2 = \sum_{r=0}^{n} \binom{n+r}{r} u_{n-r} \qquad \text{for all } n \in \mathbb{N}.$$

10.2 Solution

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References