Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2020: Homework 6

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November 13, 2020

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1 Exercise 1

1.1 Problem

Let n be a positive integer. Consider a round-robin tournament in which n players participate. ("Round-robin" means that each pair of distinct players play exactly one match against one another.) No match ends with a draw.

A player a is said to have *directly owned* a player b if a has won the match against b.

A player a is said to have *indirectly owned* a player b if there exists a player c such that a has won a match against c and c has won a match against b.

Prove that there exists a player who has (directly or indirectly) owned all other players.

1.2 Solution

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$. Assume that $a_1, a_2, \ldots, a_{2n+1}$ are $2n+1$ integers with the following property:

Weaker splitting property: If any of the first $2n$ numbers $a_1, a_2, \ldots, a_{2n}$ is removed, then the remaining $2n$ numbers (including a_{2n+1}) can be split into two equinumerous heaps with equal sum. ("Equinumerous" means that each heap contains exactly n numbers.)

Prove that all $2n+1$ numbers $a_1, a_2, \ldots, a_{2n+1}$ are equal.

2.2 Solution

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3 Exercise 3

3.1 Problem

Let X be a finite set. Let $n = |X|$. Let $f : X \to X$ be a map. Let $x \in X$. Let $p, q \in \mathbb{N}$. Prove that there exists some $k \in \{1, 2, \ldots, n\}$ such that $f^{k+p}(x) = f^{2k+q}(x)$.

3.2 Solution

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4 Exercise 4

4.1 Problem

Let X be a finite nonempty set. Let $n = |X|$. Let $f : X \to X$ be a map. Prove that $f^n(X) = f^{n-1}(X)$.

4.2 Solution

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5 Exercise 5

5.1 Problem

Let X be a set. Let $f : X \to X$ be a map. Prove the following:

(a) If some $x \in X$ and $k \in \mathbb{N}$ satisfy $f^k(x) = f^{2k}(x)$, then $f^{ik}(x) = f^k(x)$ for every positive integer i.

Now, assume that X is finite, and let $n = |X|$. Then:

(b) We have $f^{n!} = f^{2n!}$.

(c) If f is a permutation of X, then $f^{n!} = \mathrm{id}_X$.

5.2 Solution

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6 Exercise 6

6.1 Problem

Complete the solution to Exercise 6.2.3 in the notes (about periodicity of $(x_0 \% m, x_1 \% m, x_2 \% m, \ldots)$) by proving Claims 1, 2 and 3.

6.2 Solution

Let m be a positive integer. Let a and b be integers such that $b \perp m$. Let $(x_0, x_1, x_2, \ldots)$ be any (a, b)-recurrent sequence. Let M be the m-element set $\{0, 1, \ldots, m-1\}$.

For each integer z, let $\widetilde{z}$ denote the remainder $z \% m$. Thus, every integer z satisfies $\widetilde{z} = z \% m \in M$ and

$$\widetilde{z} = z \% m \equiv z \mod m \qquad (1)$$

Define a map $f : M \times M \to M \times M$ by

$$f((p, q)) = \left(q, \widetilde{aq + bp}\right) \qquad \text{for each } (p, q) \in M \times M.$$

Let ω be the pair $(\widetilde{x_0}, \widetilde{x_1}) \in M \times M$.

Now, we claim the following:

Claim 1: The map f is bijective (thus a permutation of the set $M \times M$).

Claim 2: We have $f((\widetilde{x_i}, \widetilde{x_{i+1}})) = (\widetilde{x_{i+1}}, \widetilde{x_{i+2}})$ for each $i \in \mathbb{N}$.

Claim 3: We have $(\widetilde{x_i}, \widetilde{x_{i+1}}) = f^i(\omega)$ for each $i \in \mathbb{N}$.

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7 Exercise 7

7.1 Problem

Improve Exercise 6.2.3 as follows:

Let $m > 1$ be an integer. Let a and b be integers such that $b \perp m$. Let $(x_0, x_1, x_2, \ldots)$ be any (a, b)-recurrent sequence. Prove that the sequence $(x_0 \% m, x_1 \% m, x_2 \% m, \ldots)$ is k-periodic for some $k \in \{1, 2, \ldots, m^2 - 1\}$.

7.2 Solution

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8 Exercise 8

8.1 Problem

Let $n \geq 3$ be an integer. Let $a_1, a_2, \ldots, a_{n-1}$ be any $n - 1$ integers. Assume that $n \nmid a_1 - a_2$. Prove that there exists a nonempty subset I of $\{1, 2, \ldots, n - 1\}$ such that

$$n \mid \sum_{i \in I} a_i.$$

8.2 Solution

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9 Exercise 9

9.1 Problem

For every $x \in \mathbb{R}$, we define the *fractional part* $\operatorname{frac} x$ of x to be the number $x - \lfloor x \rfloor$. Note that $0 \leq \operatorname{frac} x < 1$ for each $x \in \mathbb{R}$. Prove the following:

(a) For each $x \in \mathbb{R}$ and each positive integer n, there exists a positive integer m such that $\operatorname{frac}(mx)$ is either $< \dfrac{1}{n}$ or $> \dfrac{n-1}{n}$.

(b) For any positive real ε and any real z, there exists a positive integer m such that $|\sin(mz)| < \varepsilon$.

[**Hint:** For part **(a)**, subdivide the half-open interval $[0, 1)$ into n intervals

$$\left[\frac{0}{n}, \frac{1}{n}\right), \quad \left[\frac{1}{n}, \frac{2}{n}\right), \quad \ldots, \quad \left[\frac{n-1}{n}, \frac{n}{n}\right),$$

and argue that two of the numbers $\operatorname{frac}(0x), \operatorname{frac}(1x), \ldots, \operatorname{frac}(nx)$ must lie in one of these n intervals. If these are $\operatorname{frac}(ix)$ and $\operatorname{frac}(jx)$ (with $i < j$), then what can you say about $\operatorname{frac}(jx - ix)$?]

9.2 Solution

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10 Exercise 10

10.1 Problem

Let $R = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \right\}$. Find the smallest positive real ε such that the entire set R can be covered with 5 **closed** intervals of length ε each.

10.2 Solution

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References