Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2020: Homework 5

Darij Grinberg

November 5, 2020

---

1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$ and $a \in \mathbb{N}$. Prove that $an + 1 \mid \dbinom{(a+1)\,n}{n}$.

1.2 Solution

...

---

2 Exercise 2

2.1 Problem

Let $m \in \mathbb{Z}$. Let P be a polynomial in a single variable x. Assume that P has degree $\leq m$. Prove the following:

(a) The polynomial ΔP has degree $\leq m - 1$.

(b) The polynomial $\Delta^n P$ has degree $\leq m - n$ for each $n \in \mathbb{N}$.

(c) For each $n \in \mathbb{N}$, we have

$$\left(\Delta^n P\right)(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} P(x-k).$$

(d) For each $n \in \mathbb{N}$ satisfying $n > m$, we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} P(x-k) = 0.$$

(e) Assume that $m \geq 0$. The sequence $(P(0), P(1), P(2), \ldots)$ is $(d_1, d_2, \ldots, d_{m+1})$-recurrent, where we set $d_i = (-1)^{i-1} \binom{m+1}{i}$ for each $i \in \{1, 2, \ldots, m+1\}$.

2.2 Solution

...

---

3 Exercise 3

3.1 Problem

Let $a, b, c \in \mathbb{N}$ be such that $c \leq b$ and $a \leq b$. Simplify

$$\sum_{k=c}^{b} \frac{\binom{a}{k}}{\binom{b}{k}}.$$

3.2 Solution

...

---

4 Exercise 4

4.1 Problem

Let $q \in \mathbb{R}$ and $s \in \mathbb{R}$. Define a sequence $(a_0, a_1, a_2, \ldots)$ of reals recursively by

$$a_0 = s, \qquad \text{and}$$
$$a_n = a_{n-1}(qa_{n-1} + 2) \qquad \text{for each integer } n \geq 1.$$

Find an explicit formula for a_n.

4.2 Solution

...

---

5 Exercise 5

5.1 Problem

Define a sequence $(a_1, a_2, a_3, \ldots)$ of integers recursively by

$$a_1 = 1, \qquad a_2 = 1, \qquad a_3 = 3, \qquad \text{and}$$
$$a_n = \frac{a_{n-1}^2 - a_{n-2}^2}{a_{n-3}} \qquad \text{for each integer } n \geq 4.$$

Compute a_n explicitly (in terms of sequences we already know).

5.2 Solution

...

---

6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$. Let $\mathbb{N}_2$ denote the set $\{2, 3, 4, \ldots\}$. Prove that there is a unique lacunar subset T of $\mathbb{N}_2$ such that $n = \sum_{t \in T} f_t$.

(For example, $28 = f_3 + f_5 + f_8 = \sum_{t \in \{3,5,8\}} f_t$.)

6.2 Solution

...

---

7 Exercise 7

7.1 Problem

Solve Exercise 5.2.1 if n is not known to the lecturer. That is, find a way to construct the moments a and b in Exercise 5.2.1 in such a way that the lecturer will know that these moments have arrived the very time they arrive (rather than only in hindsight).

7.2 Solution

...

---

8 Exercise 8

8.1 Problem

Let n and k be positive integers with $k \geq 2$. Let $I_1, I_2, \ldots, I_n$ be n nonempty finite closed intervals on the real axis. Assume that for any k distinct elements $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$, at least two of the k intervals $I_{i_1}, I_{i_2}, \ldots, I_{i_k}$ intersect. Prove that there exist $k-1$ reals $a_1, a_2, \ldots, a_{k-1}$ such that each of the intervals $I_1, I_2, \ldots, I_n$ contains at least one of $a_1, a_2, \ldots, a_{k-1}$.

8.2 Solution

...

---

9 Exercise 9

Let n be a positive integer. Let $(x_0, x_1, x_2, \ldots)$ be an n-periodic sequence of reals such that $x_0 + x_1 + \cdots + x_{n-1} = 0$. Prove that there exists some $k \in \{0, 1, \ldots, n-1\}$ such that every $m \geq k$ satisfies $x_k + x_{k+1} + \cdots + x_m \geq 0$.

9.1 Solution

...

---

10 Exercise 10

10.1 Problem

A *golden pair* will mean a pair (x, y) of nonnegative integers such that $|x^2 - xy - y^2| = 1$. For example, $(3, 2)$ is a golden pair, since $|3^2 - 3 \cdot 2 - 2^2| = |-1| = 1$. Prove the following:

(a) If (x, y) is a golden pair such that $(x, y) \neq (0, 1)$, then $x - y \geq 0$.

(b) If (x, y) is a golden pair such that $(x, y) \neq (0, 1)$, then $(y, x - y)$ is a golden pair.

(c) If (x, y) is a golden pair such that $(x, y) \neq (0, 1)$ and $(x, y) \neq (1, 0)$, then $y > 0$.

(d) Find an explicit formula for all golden pairs different from $(0, 1)$.

10.2 Remark

The word "golden" in "golden pair" refers to the resemblance of the equality $\left|x^2 - xy - y^2\right| = 1$ to the equation $\varphi^2 - \varphi - 1 = 0$ satisfied by the golden ratio $\varphi \approx 1.618$. Dividing the equation $\left|x^2 - xy - y^2\right| = 1$ by y^2 yields $\left|\left(\dfrac{x}{y}\right)^2 - \dfrac{x}{y} - 1\right| = \dfrac{1}{y^2}$, which is a way of saying that $\dfrac{x}{y}$ is a close rational approximation to φ.]

10.3 Solution

...

References