Drexel University, Department of Mathematics

# Math 235: Mathematical Problem Solving, Fall 2020: Homework 4

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October 30, 2020

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## 1 Exercise 1

### 1.1 Problem

Let $u$ be a positive integer. Let $k \in \mathbb{N}$. Define a sequence $(a_0, a_1, a_2, \ldots)$ of integers by setting

$$a_n = \binom{n}{k} \% u \qquad \text{for each } n \in \mathbb{N}.$$

Show that this sequence $(a_0, a_1, a_2, \ldots)$ is $uk!$-periodic.

### 1.2 Solution

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## 2 Exercise 2

### 2.1 Problem

Let $u$ and $v$ be two integers. Let $(x_0, x_1, x_2, \ldots)$ be a $(u, v)$-recurrent sequence of integers with $x_0 = 0$.

Show that all $a, b \in \mathbb{N}$ satisfying $a \mid b$ satisfy $x_a \mid x_b$.

### 2.2 Solution

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## 3 Exercise 3

### 3.1 Problem

Generalize the generalized Cassini identity[1] further, to a claim about two $(a, b)$-recurrent sequences $(x_0, x_1, x_2, \ldots)$ and $(y_0, y_1, y_2, \ldots)$.

[**Hint:** The left hand side will be $x_{n+1}y_{n-1} - x_n y_n$.]

### 3.2 Solution

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## 4 Exercise 4

### 4.1 Problem

Let $a$ be any number. Let $(x_0, x_1, x_2, \ldots)$ and $(y_0, y_1, y_2, \ldots)$ be two $(a, 1)$-recurrent sequences of numbers with $x_0 = 0$. (We don't require anything of $y_0$.) Let $n, m \in \mathbb{N}$ satisfy $n \geq m$. Prove that

$$x_{n-m}y_{n+m} = x_n y_n - (-1)^{n+m} x_m y_m.$$

### 4.2 Solution

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## 5 Exercise 5

### 5.1 Problem

Let $u$ and $v$ be two integers such that $u \perp v$. Let $(x_0, x_1, x_2, \ldots)$ be a $(u, v)$-recurrent sequence of integers with $x_0 = 0$ and $x_1 = 1$. Show that all $a, b \in \mathbb{N}$ satisfy $\gcd(x_a, x_b) = \left|x_{\gcd(a,b)}\right|$.

[1] which says that $x_{n+1}x_{n-1} - x_n^2 = (-b)^{n-1}(x_2 x_0 - x_1^2)$ for any $(a, b)$-recurrent sequence $(x_0, x_1, x_2, \ldots)$

### 5.2 Solution

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## 6 Exercise 6

### 6.1 Problem

Let $n \in \mathbb{N}$. Prove that there exists some $m \in \mathbb{N}$ such that $\left(\sqrt{2}-1\right)^n = \sqrt{m+1} - \sqrt{m}$.

### 6.2 Solution

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## 7 Exercise 7

### 7.1 Problem

A sequence $(a_0, a_1, a_2, \ldots)$ of numbers is defined recursively by $a_0 = -1$ and $a_1 = 0$ and $a_n = a_{n-1}^2 - n^2 a_{n-2} - 1$ for all $n \geq 1$. Find $a_{100}$.

### 7.2 Solution

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## 8 Exercise 8

### 8.1 Problem

Let $n \in \mathbb{N}$. Prove that $\left\lfloor \left(1+\sqrt{2}\right)^n \right\rfloor$ is even if and only if $n$ is odd.

### 8.2 Solution

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# 9 Exercise 9

## 9.1 Problem

Let $m$ and $k$ be two positive integers such that $m \mid k+1$. Define a sequence $(a_0, a_1, a_2, \ldots)$ of positive integers recursively by

$$a_0 = 1, \qquad a_1 = 1, \qquad a_2 = m \qquad \text{and}$$
$$a_n = \frac{k + a_{n-1}a_{n-2}}{a_{n-3}} \qquad \text{for each } n \geq 3.$$

Prove that $a_n$ is an integer for each $n \in \mathbb{N}$.

[**Hint:** Prove that each of the two subsequences $(a_0, a_2, a_4, a_6, \ldots)$ and $(a_1, a_3, a_5, a_7, \ldots)$ is $(a, b)$-recurrent for some integers $a$ and $b$ (but not the same integers for both subsequences!).]

## 9.2 Solution

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# 10 Exercise 10

## 10.1 Problem

Let $(a, a+d, a+2d, a+3d, \ldots)$ be any (infinite) arithmetic progression with $d \neq 0$. Prove that this arithmetic progression contains an infinite geometric progression as a subsequence (i.e., there is an infinite strictly increasing sequence $(i_0, i_1, i_2, \ldots)$ of nonnegative integers such that $(a + i_0 d, a + i_1 d, a + i_2 d, \ldots)$ is a geometric progression) if and only if $\frac{a}{d} \in \mathbb{Q}$.

## 10.2 Solution

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# References