Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2020: Homework 3

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1 Exercise 1

1.1 Problem

Let $(a_0, a_1, a_2, \ldots)$ be a sequence of integers defined recursively by

$$a_0 = 0, \qquad a_1 = 1, \qquad \text{and}$$
$$a_n = 1 + a_{n-1}a_{n-2} \qquad \text{for each integer } n \geq 2.$$

Prove the following:

(a) For any $k \in \mathbb{N}$ and $n \in \mathbb{N}$, we have $a_{k+n} \equiv a_k \mod a_n$.

(b) If $u, v \in \mathbb{N}$ satisfy $u \mid v$, then $a_u \mid a_v$.

(c) For any $n, m \in \mathbb{N}$, we have $\gcd(a_n, a_m) = a_{\gcd(n,m)}$.

1.2 Solution

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2 Exercise 2

2.1 Problem

For any positive integer n, we let $d(n)$ denote the number of all positive divisors of n. (For example, $d(6) = 4$.)

Let n be a positive integer. Prove that

$$d(1) + d(2) + \cdots + d(n) = \left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor .$$

2.2 Solution

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3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$.

(a) Simplify $\sum\limits_{k=0}^{n} k \dbinom{n}{k}$. ("Simplify" means "get rid of the $\sum$ sign".)

(b) Simplify $\sum\limits_{k=0}^{n} \dbinom{2n}{k}$.

3.2 Solution

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Let $x_1, x_2, \ldots, x_n$ be any numbers, and let y be a further number. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

(a) Prove that every $m \in \{0, 1, \ldots, n-1\}$ satisfies

$$\sum_{I \subseteq [n]} (-1)^{n-|I|} \left(y + \sum_{i \in I} x_i \right)^m = 0.$$

(b) Prove that

$$\sum_{I \subseteq [n]} (-1)^{n-|I|} \left(y + \sum_{i \in I} x_i \right)^n = n! x_1 x_2 \cdots x_n .$$

[**Remark:** For $n = 2$, the statement of part **(b)** says that

$$(y + x_1 + x_2)^2 - (y + x_1)^2 - (y + x_2)^2 + y^2 = 2! x_1 x_2.$$

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4.2 Solution

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5 Exercise 5

5.1 Problem

[Problem invalid: Apparently there is no closed-form answer. Sorry!]

Let $a \in \mathbb{R}$ and $b, c \in \mathbb{N}$ be such that $c \leq b$. Simplify

$$\sum_{k=c}^{b} (-1)^k \frac{\dbinom{a}{k}}{\dbinom{b}{k}}.$$

5.2 Solution

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6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$ and $x \in \mathbb{R} \setminus \{1, -1\}$. For each $i \in \mathbb{N}$, set $y_i = 1 - x^i$. Prove that

$$\sum_{k=0}^{n-1} \frac{y_n y_{n-1} \cdots y_{n-k}}{y_{k+1}} = n.$$

6.2 Solution

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7 Exercise 7

7.1 Problem

Define a sequence $(a_0, a_1, a_2, \ldots)$ of integers recursively by

$$a_0 = 0, \qquad a_1 = 1, \qquad \text{and}$$
$$a_n = n\left(a_{n-1} + (n-1)\, a_{n-2}\right) \qquad \text{for each integer } n \geq 2.$$

Compute a_n explicitly (in terms of sequences we already know).

7.2 Solution

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8 Exercise 8

8.1 Problem

Let a, b, u and v be four reals.

We define two sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ of reals recursively by setting

$$a_0 = a \qquad \text{and} \qquad b_0 = b$$

and

$$a_n = ua_{n-1} + vb_{n-1} \qquad \text{and} \qquad b_n = ub_{n-1} + va_{n-1}$$

for each $n \geq 1$.

Find explicit formulas for a_n and b_n.

8.2 Solution

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9 Exercise 9

9.1 Problem

Let $(a_0, a_1, a_2, \ldots)$ be the unique weakly increasing sequence of positive integers that contains each positive integer i exactly i times. Thus,

$$(a_0, a_1, a_2, \ldots) = (1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots).$$

Prove that

$$a_n = \left\lfloor \frac{1}{2}\sqrt{8n+1} + \frac{1}{2} \right\rfloor \qquad \text{for each } n \in \mathbb{N}.$$

9.2 Solution

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10 Exercise 10

10.1 Problem

Let a and b be two positive reals. Let $f : \mathbb{R} \to \mathbb{R}$ be an a-periodic function. Let $g : \mathbb{R} \to \mathbb{R}$ be a b-periodic function. Consider the function $f + g : \mathbb{R} \to \mathbb{R}$ (which sends each $x \in \mathbb{R}$ to $(f+g)(x)$).

(a) If $a/b \in \mathbb{Q}$, then prove that $f + g$ is again a periodic function.

(b) Show that $f + g$ need not be periodic if $a/b \notin \mathbb{Q}$. (Feel free to interpret this either as "Find an example where $f + g$ is not periodic for **some** pair of a and b satisfying $a/b \notin \mathbb{Q}$" or as "Find an example where $f + g$ is not periodic for **every** pair of a and b satisfying $a/b \notin \mathbb{Q}$".)

10.2 Solution

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11 Exercise 11

11.1 Problem

Let $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Prove that

$$\gcd\left(\dbinom{n-1}{k-1}, \dbinom{n}{k+1}, \dbinom{n+1}{k}\right) = \gcd\left(\dbinom{n-1}{k}, \dbinom{n}{k-1}, \dbinom{n+1}{k+1}\right).$$

11.2 Solution

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References