Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2020: Homework 2

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1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$. Let $a_1, a_2, \ldots, a_n$ be n **odd** integers. Prove that

$$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 \equiv n \mod 4.$$

1.2 Solution

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2 Exercise 2

2.1 Problem

Let a and b be two coprime positive integers.

(a) Prove that there do **not** exist any positive integers x and y satisfying $ab = xa + yb$.

(b) Prove that there do **not** exist any $x, y \in \mathbb{N}$ satisfying $ab - a - b = xa + yb$.

2.2 Solution

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3 Exercise 3

3.1 Problem

Let n and m be two coprime positive integers. Let $u \in \mathbb{Z}$. Prove that

$$(u^n - 1)(u^m - 1) \mid (u - 1)(u^{mn} - 1).$$

3.2 Solution

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4 Exercise 4

4.1 Problem

Let $n, m \in \mathbb{N}$ satisfy $n > 0$. Prove the following:

(a) We have $\dfrac{m}{n}\dbinom{n}{m} = \dbinom{n-1}{m-1}$.

(b) We have $\dfrac{\gcd(n,m)}{n}\dbinom{n}{m} \in \mathbb{Z}$.

4.2 Solution

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5 Exercise 5

5.1 Problem

Let n, i and j be positive integers such that $i < n$ and $j < n$. Prove that

$$\gcd\left(\binom{n}{i}, \binom{n}{j}\right) > 1.$$

5.2 Solution

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6 Exercise 6

6.1 Problem

Let n be a positive integer. We let $\phi(n)$ denote the number of all $i \in \{1, 2, \ldots, n\}$ satisfying $i \perp n$. (For example, $\phi(12) = 4$, because there are exactly 4 numbers $i \in \{1, 2, \ldots, 12\}$ satisfying $i \perp 12$: namely, 1, 5, 7 and 11.)

(a) Prove that $\phi(n)$ is even if $n > 2$.

(b) Prove that the sum of all $i \in \{1, 2, \ldots, n\}$ satisfying $i \perp n$ equals $\frac{1}{2} n\phi(n)$ if $n > 1$.

[**Remark:** The function $\phi : \{1, 2, 3, \ldots\} \to \mathbb{N}$ that sends each positive integer n to $\phi(n)$ is known as the *Euler totient function* (or the *phi-function*). Here is a table of its first few values:

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$\phi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12

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Can you spot any patterns?]

6.2 Solution

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7 Exercise 7

7.1 Problem

Let $(f_0, f_1, f_2, \ldots)$ be the Fibonacci sequence. Find $\sum\limits_{k=2}^{\infty} \frac{f_k}{f_{k-1} f_{k+1}}$.

7.2 Solution

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8 Exercise 8

8.1 Problem

(a) Prove that

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1} \qquad \text{for each } n \in \mathbb{N} \text{ and } k \in \mathbb{N}.$$

(b) Prove that

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m} \qquad \text{for each } n \in \mathbb{C} \text{ and } m \in \mathbb{N}.$$

8.2 Solution

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9 Exercise 9

9.1 Problem

Let $(f_0, f_1, f_2, \ldots)$ be the Fibonacci sequence. Prove that

$$2^{n-1} \cdot f_n = \sum_{k=0}^{n} \binom{n}{2k+1} 5^k \qquad \text{for each } n \in \mathbb{N}.$$

9.2 Solution

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10 Exercise 10

10.1 Problem

Let $(f_0, f_1, f_2, \ldots)$ be the Fibonacci sequence. For this exercise, we also set $f_{-1} = 1$.

For any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, define the rational number $\binom{n}{k}_F$ (a slight variation on the corresponding binomial coefficient) by

$$\binom{n}{k}_F = \begin{cases} \dfrac{f_n f_{n-1} \cdots f_{n-k+1}}{f_k f_{k-1} \cdots f_1}, & \text{if } n \geq k \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Prove that $\binom{n}{k}_F = \binom{n}{n-k}_F$ for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$.

(b) Let n be a positive integer, and let $k \in \mathbb{N}$ be such that $n \geq k$. Prove that

$$\binom{n}{k}_F = f_{k+1}\binom{n-1}{k}_F + f_{n-k-1}\binom{n-1}{k-1}_F.$$

(c) Prove that $\binom{n}{k}_F \in \mathbb{N}$ for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$.

10.2 Solution

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References