Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2020: Homework 0

Darij Grinberg

September 28, 2020

---

1 Exercise 1

1.1 Problem

Let n be an even positive integer. Find a $q \in \{1, 2, \ldots, 2n\}$ such that

$$\frac{1! \cdot 2! \cdot \cdots \cdot (2n)!}{q!} \text{ is a perfect square.}$$

1.2 Solution

...

---

2 Exercise 2

2.1 Problem

Define a sequence $(t_0, t_1, t_2, \ldots)$ of positive rational numbers recursively by setting

$$t_0 = 1, \qquad t_1 = 1, \qquad t_2 = 1, \qquad \text{and}$$
$$t_n = \frac{1 + t_{n-1}t_{n-2}}{t_{n-3}} \qquad \text{for each } n \geq 3.$$

(For example, $t_3 = \dfrac{1 + t_2 t_1}{t_0} = \dfrac{1 + 1 \cdot 1}{1} = 2$ and $t_4 = \dfrac{1 + t_3 t_2}{t_1} = \dfrac{1 + 2 \cdot 1}{1} = 3$.)

(a) Prove that $t_{n+2} = 4t_n - t_{n-2}$ for each $n \geq 2$.

(b) Prove that t_n is a positive integer for each integer $n \geq 0$.

2.2 Solution

...

---

3 Exercise 3

3.1 Problem

Let $x \in \mathbb{R}$ and let n be a positive integer. Prove that

$$\sum_{k=0}^{n-1} \left\lfloor x + \frac{k}{n} \right\rfloor = \lfloor nx \rfloor .$$

3.2 Solution

...

---

4 Exercise 4

4.1 Problem

Let a, b, c, n be positive integers such that $a \mid b^n$ and $b \mid c^n$ and $c \mid a^n$. Prove that $abc \mid (a + b + c)^{n^2+n+1}$.

4.2 Solution

...

---

5 Exercise 5

5.1 Problem

A mountain ridge has the form of a (finite) line segment, bordered on each end by a cliff. Several (finitely many) lemmings are walking along the ridge, with equal speeds (but not necessarily in the same direction). Whenever two lemmings meet, they "bounce off" one another, keeping their respective speeds but reversing their directions. Whenever a lemming arrives at an endpoint of the ridge, it falls off the cliff. Prove that eventually, all lemmings will fall off the cliff.

[**Example:** Here is a possible lemming configuration:

$$\overrightarrow{1} \quad \overleftarrow{2} \quad\quad \overleftarrow{3} \quad\quad\quad\quad \overrightarrow{4} \quad\quad\quad \overleftarrow{5}$$

(with $1, 2, 3, 4, 5$ being the lemmings, and the arrows signifying their walking directions). The first two lemmings to meet here will be 1 and 2, after which they both change their directions:

$$\overleftarrow{1}\overrightarrow{2} \quad\quad \overleftarrow{3} \quad\quad\quad\quad \overrightarrow{4} \quad\quad \overleftarrow{5}$$

Now lemming 1 is on its way to the cliff, which it will reach without interference from other lemmings.]

5.2 Solution

...

---

6 Exercise 6

6.1 Problem

You have 20 white socks and 20 black socks hanging on a clothesline, in some order.

(a) Prove that you can pick 10 consecutive socks from the clothesline such that 5 of them are black and the other 5 white. (You can call such a pick *color-balanced*.)

(b) Prove the same if the number 20 is replaced by 12 (so you have 12 white and 12 black socks).

[**Example:** If the number 20 is replaced by 7, then the claim does not hold. For example, the configuration $B^3W^7B^4$ (standing for "3 black socks, followed by 7 white socks, followed by 4 white socks") has no color-balanced pick of 10 consecutive socks.]

6.2 Solution

...

---

7 Exercise 7

7.1 Problem

Factor the polynomial

$$bc(b-c)(b+c) + ca(c-a)(c+a) + ab(a-b)(a+b)$$

into a product of four linear (i.e., degree-1) polynomials.

7.2 Solution

...

---

8 Exercise 8

8.1 Problem

(a) Let a, b, c be three nonnegative reals. Prove that

$$|ca - ab| + |ab - bc| + |bc - ca| \leq \left|b^2 - c^2\right| + \left|c^2 - a^2\right| + \left|a^2 - b^2\right|.$$

(b) Is this still true if the word "nonnegative" is omitted?

8.2 Solution

...

---

9 Exercise 9

9.1 Problem

Let $n \geq 1$. Let $a_1, a_2, \ldots, a_n$ be any n integers. Prove that there exist some $p, q \in \{1, 2, \ldots, n\}$ with $p \leq q$ and $n \mid a_p + a_{p+1} + \cdots + a_q$.

9.2 Solution

...

---

10 Exercise 10

10.1 Problem

Briefly review the problems above: Which ones did you like? Which ones did you not like? Why? How long did they take you? Which parts did you get stuck on? Did you learn anything from solving (or trying to solve) them? If you knew the solution already (nothing wrong with that!), where did you learn it? (No need to rate every exercise; just say some words about some 4 of them.)

10.2 Solution

...

References