Math 4707 Spring 2018 (Darij Grinberg): homework set 4 with solutions

Contents

Recall the following:

- We have $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- If $n \in \mathbb{N}$, then $[n]$ denotes the n-element set $\{1, 2, \ldots, n\}$.
- For each $n \in \mathbb{N}$, we let S_n denote the set of all permutations of $[n]$.

0.1. Permutations σ with $\sigma(2) = \sigma(1) + 1$

Exercise 1. Let $n \geq 2$ be an integer. Prove that there are precisely $(n-1)!$ permutations $\sigma \in S_n$ satisfying $\sigma(2) = \sigma(1) + 1$.

Solution to Exercise 1 (sketched). The following method constructs every permutation $\sigma \in S_n$ satisfying $\sigma(2) = \sigma(1) + 1$:

- First, choose the value $\sigma(1)$. This value must belong to the set $[n-1]$ (because we want our σ to satisfy $\sigma(2) = \sigma(1) + 1$, whence $\sigma(1) + 1 = \sigma(2) \leq n$ (since $\sigma(2) \in [n]$) and therefore $\sigma(1) \leq n-1$). Thus, there are $n-1$ choices at this step.
- Then, the value $\sigma(2)$ is uniquely determined by the equality $\sigma(2) = \sigma(1) + 1$. Note that $\sigma(1)$ and $\sigma(2)$ are thus distinct.
- Now, choose the remaining $n-2$ values $\sigma(3), \sigma(4), \ldots, \sigma(n)$ of σ. These $n-2$ values must be distinct from each other and from the already chosen two values $\sigma(1)$ and $\sigma(2)$ (since we want σ to be a permutation); thus, we are choosing $n-2$ distinct values from the $(n-2)$-element set $[n] \setminus \{\sigma(1), \sigma(2)\}$ at this step. There are clearly $(n-2)!$ ways to do that.

The total number of possible ways to perform this method is thus $(n-1) \cdot (n-2)! = (n-1)!$. Hence, there are precisely $(n-1)!$ permutations $\sigma \in S_n$ satisfying $\sigma(2) = \sigma(1) + 1$. This solves Exercise 1. □

0.2. An introduction to rook theory

Definition 0.1. For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we let $x^{\underline{n}}$ denote the "n-th lower factorial of x"; this is the real number $x(x-1)\cdots(x-n+1)$. (Thus, $x^{\underline{n}} = n! \cdot \dbinom{x}{n}$.)

For example, $x^{\underline{0}} = 1$, $x^{\underline{1}} = x$, $x^{\underline{2}} = x(x-1)$, etc.[1].

Exercise 2. Let $n \in \mathbb{N}$. Prove the following:

(a) If $(a_0, a_1, \ldots, a_n)$ is an $(n+1)$-tuple of rational numbers such that each $x \in \{0, 1, \ldots, n\}$ satisfies

$$\sum_{k=0}^{n} a_k x^{\underline{k}} = 0, \tag{1}$$

then $(a_0, a_1, \ldots, a_n) = (0, 0, \ldots, 0)$.

(b) If $(a_0, a_1, \ldots, a_n)$ and $(b_0, b_1, \ldots, b_n)$ are two $(n+1)$-tuples of rational numbers such that each $x \in \{0, 1, \ldots, n\}$ satisfies

$$\sum_{k=0}^{n} a_k x^{\underline{k}} = \sum_{k=0}^{n} b_k x^{\underline{k}}, \tag{2}$$

then $(a_0, a_1, \ldots, a_n) = (b_0, b_1, \ldots, b_n)$.

[**Hint:** In terms of linear algebra, part **(a)** is saying that the $n+1$ vectors $\left(0^{\underline{k}}, 1^{\underline{k}}, \ldots, n^{\underline{k}}\right)^T \in \mathbb{Q}^{n+1}$ for $k \in \{0, 1, \ldots, n\}$ are linearly independent. You may find this useful or not; the exercise has a fully elementary solution.]

Solution to Exercise 2. Let us first observe that

$$j^{\underline{k}} = 0 \qquad \text{for any } j \in \mathbb{N} \text{ and } k \in \mathbb{N} \text{ satisfying } k \geq j+1. \tag{3}$$

[*Proof of (3):* Let $j \in \mathbb{N}$ and $k \in \mathbb{N}$ be such that $k \geq j+1$. Hence, $j+1 \leq k$, so

[1] For all LaTeX users: $x^{\underline{n}}$ is "`x^{\underline{n}}`". Feel free to create a macro for this, e.g., by putting the following line into the header of your TeX file (along with the other "VARIOUS USEFUL COMMANDS"):

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\newcommand{\lf}[2]{{#1}^{\underline{#2}}}
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Then, you can use "`\lf{x}{n}`" to obtain "$x^{\underline{n}}$".

that $j \leq k-1$ and thus $j \in \{0,1,\ldots,k-1\}$ (since $j \in \mathbb{N}$). The definition of $j^{\underline{k}}$ yields

$$\begin{aligned} j^{\underline{k}} &= j(j-1)\cdots(j-k+1) = \prod_{u=0}^{k-1}(j-u) = \prod_{u\in\{0,1,\ldots,k-1\}}(j-u) \\ &= \underbrace{(j-j)}_{=0} \prod_{\substack{u\in\{0,1,\ldots,k-1\};\\ u\neq j}}(j-u) \\ &\qquad \left(\begin{array}{c}\text{here, we have split off the factor for } u=j \text{ from}\\ \text{the product, since } j\in\{0,1,\ldots,k-1\}\end{array}\right) \\ &= 0 \prod_{\substack{u\in\{0,1,\ldots,k-1\};\\ u\neq j}}(j-u) = 0. \end{aligned}$$

This proves (3).]

(a) Let $(a_0, a_1, \ldots, a_n)$ be an $(n+1)$-tuple of rational numbers such that each $x \in \{0,1,\ldots,n\}$ satisfies (1). We must prove that $(a_0, a_1, \ldots, a_n) = (0,0,\ldots,0)$.

We claim that

$$a_k = 0 \qquad \text{for each } k \in \{0,1,\ldots,n\}. \tag{4}$$

[*Proof of (4):* We shall prove (4) by strong induction on k:

Induction step: Let $j \in \{0,1,\ldots,n\}$. Assume that (4) holds whenever $k < j$. We must then prove that (4) holds for $k = j$.

We have assumed that (4) holds whenever $k < j$. In other words, we have

$$a_k = 0 \qquad \text{for each } k \in \{0,1,\ldots,n\} \text{ satisfying } k < j. \tag{5}$$

From $j \in \{0,1,\ldots,n\}$, we obtain $0 \leq j \leq n$. Also, $j-1 \leq j \leq n$, so that $\{0,1,\ldots,j-1\} \subseteq \{0,1,\ldots,n\}$. The definition of $j^{\underline{j}}$ yields $j^{\underline{j}} = j(j-1)\cdots(j-j+1) = j(j-1)\cdots 1 = j!$.

But we have assumed that each $x \in \{0,1,\ldots,n\}$ satisfies (1). Applying this to $x = j$, we thus conclude that

$$\sum_{k=0}^{n} a_k j^{\underline{k}} = 0.$$

Hence,

$$\begin{aligned}
0 &= \sum_{k=0}^{n} a_k j^{\underline{k}} = \sum_{k=0}^{j} a_k j^{\underline{k}} + \sum_{k=j+1}^{n} a_k \underbrace{j^{\underline{k}}}_{\substack{=0 \\ \text{(by (3)} \\ \text{(since } j \geq k+1))}} \\
&\qquad \left(\text{here, we have split the sum at } k = j \text{ (since } 0 \leq j \leq n)\right) \\
&= \sum_{k=0}^{j} a_k j^{\underline{k}} + \underbrace{\sum_{k=j+1}^{n} a_k 0}_{=0} = \sum_{k=0}^{j} a_k j^{\underline{k}} \\
&= \sum_{k=0}^{j-1} \underbrace{a_k}_{\substack{=0 \\ \text{(by (5)} \\ \text{(since } k \in \{0,1,\ldots,j-1\} \subseteq \{0,1,\ldots,n\} \text{ and } k \leq j-1 < j))}} j^{\underline{k}} + a_j \underbrace{j^{\underline{j}}}_{=j!} \\
&\qquad \left(\text{here, we have split off the addend for } k = j \text{ from the sum}\right) \\
&= \underbrace{\sum_{k=0}^{j-1} 0 j^{\underline{k}}}_{=0} + a_j j! = a_j j!.
\end{aligned}$$

Dividing this equality by $j!$, we obtain $0 = a_j$ (since $j!$ is nonzero). In other words, $a_j = 0$. In other words, (4) holds for $k = j$. This completes the induction step. Thus, (4) is proven by strong induction.]

Now, (4) immediately yields $(a_0, a_1, \ldots, a_n) = (0, 0, \ldots, 0)$. This solves Exercise 2 **(a)**.

(b) Let $(a_0, a_1, \ldots, a_n)$ and $(b_0, b_1, \ldots, b_n)$ be two $(n+1)$-tuples of rational numbers such that each $x \in \{0, 1, \ldots, n\}$ satisfies (2). We must prove that $(a_0, a_1, \ldots, a_n) = (b_0, b_1, \ldots, b_n)$.

Clearly, $(a_0 - b_0, a_1 - b_1, \ldots, a_n - b_n)$ is an $(n+1)$-tuple of rational numbers (since $(a_0, a_1, \ldots, a_n)$ and $(b_0, b_1, \ldots, b_n)$ are two $(n+1)$-tuples of rational numbers). Moreover, each $x \in \{0, 1, \ldots, n\}$ satisfies

$$\sum_{k=0}^{n} \underbrace{(a_k - b_k) x^{\underline{k}}}_{=a_k x^{\underline{k}} - b_k x^{\underline{k}}} = \sum_{k=0}^{n} \left(a_k x^{\underline{k}} - b_k x^{\underline{k}}\right) = \underbrace{\sum_{k=0}^{n} a_k x^{\underline{k}}}_{\substack{= \sum_{k=0}^{n} b_k x^{\underline{k}} \\ \text{(by (2))}}} - \sum_{k=0}^{n} b_k x^{\underline{k}} = \sum_{k=0}^{n} b_k x^{\underline{k}} - \sum_{k=0}^{n} b_k x^{\underline{k}} = 0.$$

Hence, Exercise 2 **(a)** (applied to $a_k - b_k$ instead of a_k) yields $(a_0 - b_0, a_1 - b_1, \ldots, a_n - b_n) = (0, 0, \ldots, 0)$. In other words, $a_k - b_k = 0$ for each $k \in \{0, 1, \ldots, n\}$. In other words, $a_k = b_k$ for each $k \in \{0, 1, \ldots, n\}$. In other words, $(a_0, a_1, \ldots, a_n) = (b_0, b_1, \ldots, b_n)$. This solves Exercise 2 **(b)**. □

Remark 0.2. There are other solutions to Exercise 2, using linear algebra or the properties of polynomials. But I think the above solution is the easiest one. Of course, the analogue of Exercise 2 in which the lower factorials $x^{\underline{k}}$ are replaced by (usual) powers x^k is well-known, and follows (e.g.) from Lagrange interpolation.

In the following, we will consider each pair $(i, j) \in \mathbb{Z}^2$ of two integers as a square on an (infinite) chessboard; we say that it lies in *row* i and in *column* j. A *rook placement* shall mean a subset X of $\mathbb{Z}^2$ such that any two distinct elements of X lie in different rows and in different columns[2]. (The idea behind this name is that if we place rooks into the squares $(i, j) \in X$, then no two rooks will attack each other.) For example, $\{(1,3), (2,2), (3,7)\}$ is a rook placement, whereas $\{(1,3), (2,2), (7,3)\}$ is not (since the distinct squares $(1,3)$ and $(7,3)$ lie in the same column). If X is a rook placement, then the elements of X are called the *rooks* of X.

Now, fix $n \in \mathbb{N}$. If $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ is an n-tuple of nonnegative integers, then we define the $\mathbf{u}$*-board* $D(\mathbf{u})$ to be the set

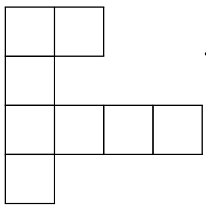
$$\{(i, j) \mid i \in [n] \text{ and } j \in [u_i]\}.$$

We visually represent this set as an "irregular chessboard" consisting of n left-aligned rows[3], where the i-th row consists of u_i boxes (which occupy columns $1, 2, \ldots, u_i$).

For example, if $n = 4$ and $\mathbf{u} = (2, 1, 4, 1)$, then $D(\mathbf{u})$ is the set

$$\{(1,1), (1,2), (2,1), (3,1), (3,2), (3,3), (3,4), (4,1)\},$$

and is visually represented as the "irregular chessboard"



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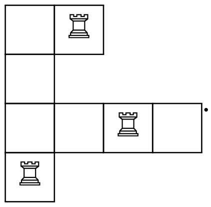
The set (or "irregular chessboard") $D(\mathbf{u})$ is also known as the *Young diagram* (or *Ferrers diagram*) of $\mathbf{u}$; we have briefly seen its use in the proof of Proposition 4.13 on March 21st.

If $\mathbf{u} \in \mathbb{N}^n$, then a *rook placement in* $D(\mathbf{u})$ means a subset X of $D(\mathbf{u})$ that is a rook placement. For example, if $n = 4$ and $\mathbf{u} = (2, 1, 4, 1)$, then $\{(1,2), (3,3), (4,1)\}$ is a rook placement in $D(\mathbf{u})$. We represent this rook placement by putting rooks (i.e.,

[2] In other words, a rook placement means a subset X of $\mathbb{Z}^2$ such that any two distinct elements (i, j) and (i', j') of X satisfy $i \neq i'$ and $j \neq j'$.

[3] or "ranks", to use the terminology of chess (but we label them $1, 2, \ldots, n$ from top to bottom, not from bottom to top as on an actual chessboard)

♖ symbols) into the squares that belong to it; so we get



Note that the empty set $\varnothing$ is always a rook placement in $D(\mathbf{u})$; so is any 1-element subset of $D(\mathbf{u})$.

If $\mathbf{u} \in \mathbb{N}^n$ and $k \in \mathbb{Z}$, then we let $R_k(\mathbf{u})$ denote the number of all rook placements in $D(\mathbf{u})$ of size k. (In terms of the picture, "size k" means that it contains exactly k rooks.) For example, if $n = 3$ and $\mathbf{u} = (2, 1, 3)$, then

$$R_0(\mathbf{u}) = 1; \qquad R_1(\mathbf{u}) = 6; \qquad R_2(\mathbf{u}) = 7;$$
$$R_3(\mathbf{u}) = 1; \qquad R_k(\mathbf{u}) = 0 \text{ for all } k \geq 4;$$
$$R_k(\mathbf{u}) = 0 \text{ for all } k < 0.$$

It is easy to see that if $\mathbf{u} = \left(\underbrace{u, u, \ldots, u}_{n \text{ times}}\right)$ for some $u \in \mathbb{N}$, then

$$R_k(\mathbf{u}) = \binom{n}{k}\binom{u}{k} k! \qquad \text{for each } k \in \mathbb{N}. \tag{6}$$

(Indeed, in order to place k non-attacking rooks in $D(\mathbf{u})$, we first choose the k rows they will occupy, then the k columns they will occupy, and finally an appropriate permutation of $[k]$ that will determine which column has a rook in which row.) For other n-tuples $\mathbf{u}$, finding $R_k(\mathbf{u})$ is harder.

For example, if you try the same reasoning for $\mathbf{u} = (2n-1, 2n-3, \ldots, 5, 3, 1)$, then you still have $\binom{n}{k}$ choices for the k rows occupied by rooks; but the number of options in the following steps will depend on the specific k rows you have chosen (the lower the rows, the fewer options). This way, you get the formula

$$R_k(2n-1, 2n-3, \ldots, 5, 3, 1) = \sum_{1 \leq s_1 < s_2 < \cdots < s_k \leq n} (s_k - 0)(s_{k-1} - 1)(s_{k-2} - 2) \cdots (s_1 - (k-1)),$$

which is a far cry from the simplicity of (6). But there is a simple formula, which we'll see in Corollary 0.4!

We can restrict ourselves to n-tuples $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ satisfying $u_1 \geq u_2 \geq \cdots \geq u_n$, because switching some of the entries of $\mathbf{u}$ does not change the values of $R_k(\mathbf{u})$.

We have

$$R_k(\mathbf{u}) = 0 \qquad \text{for all integers } k > n, \tag{7}$$

because each of the n rows $1, 2, \ldots, n$ contains at most one rook. For $k = n$, we have a neat formula:

Proposition 0.3. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ be such that $u_1 \geq u_2 \geq \cdots \geq u_n$. Then,

$$R_n(\mathbf{u}) = \prod_{i=0}^{n-1} (u_{n-i} - i).$$

Proof of Proposition 0.3. Each square in $D(\mathbf{u})$ belongs to one of the n rows $1, 2, \ldots, n$. In a rook placement, each row contains at most one rook. Thus, any rook placement in $D(\mathbf{u})$ contains at most n rooks, and the only way it can contain n rooks is if it contains exactly one rook in each of the n rows $1, 2, \ldots, n$.

Hence, a rook placement in $D(\mathbf{u})$ of size n is the same as a rook placement in $D(\mathbf{u})$ that contains a rook in each of the n rows $1, 2, \ldots, n$. We can construct such a rook placement by the following algorithm:

- First, we place the rook in row n. This rook must lie in one of the u_n columns $1, 2, \ldots, u_n$; so we have u_n options.
- Then, we place the rook in row $n-1$. This rook must lie in one of the u_{n-1} columns $1, 2, \ldots, u_{n-1}$, but **not** in the column that contains the previous rook; so we have $u_{n-1} - 1$ options.
- Then, we place the rook in row $n-2$. This rook must lie in one of the u_{n-2} columns $1, 2, \ldots, u_{n-2}$, but **not** in either of the two columns that contain the previous rooks; so we have $u_{n-2} - 2$ options[4].
- And so on.

Thus, the total number of choices (and therefore the total number of rook placements in $D(\mathbf{u})$ of size n) is

$$u_n (u_{n-1} - 1)(u_{n-2} - 2) \cdots \left(u_{n-(n-1)} - (n-1)\right)$$
$$= (u_n - 0)(u_{n-1} - 1)(u_{n-2} - 2) \cdots (u_1 - (n-1)) = \prod_{i=0}^{n-1} (u_{n-i} - i).$$

Since the total number of rook placements in $D(\mathbf{u})$ of size n has been denoted by $R_n(\mathbf{u})$, we thus conclude that

$$R_n(\mathbf{u}) = \prod_{i=0}^{n-1} (u_{n-i} - i).$$

This proves Proposition 0.3.

[Are you wondering where we have used the condition $u_1 \geq u_2 \geq \cdots \geq u_n$?

[4]Why exactly $u_{n-2} - 2$? Because the two previous rooks lie in two **different** columns (due to the way we placed the second rook), and thus there are $u_{n-2} - 2$ of the u_{n-2} columns $1, 2, \ldots, u_{n-2}$ left for us to place our third rook in.

Answer:

Answer: Consider the algorithm above. When we placed the rook in row $n-2$, we argued that we had $u_{n-2}-2$ options, because the rook could be placed in any of the u_{n-2} columns $1,2,\ldots,u_{n-2}$ except for the two columns that contain the previous rooks. This tacitly relied on the fact that the two "forbidden" columns (i.e., the two columns that contain the previous rooks) are among the u_{n-2} columns $1,2,\ldots,u_{n-2}$ (if this was not the case, then we would have more than $u_{n-2}-2$ options for our rook). This fact is true because the first "forbidden" column is among the columns $1,2,\ldots,u_n$ and therefore also among the columns $1,2,\ldots,u_{n-2}$ (since $u_{n-2} \geq u_n$), and similarly for the second "forbidden" column.

] $\square$

Exercise 3. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ be such that $u_1 \geq u_2 \geq \cdots \geq u_n$.

(a) For each $x \in \mathbb{N}$, define an n-tuple $\mathbf{u} + x \in \mathbb{N}^n$ by $\mathbf{u} + x = (u_1 + x, u_2 + x, \ldots, u_n + x)$. Prove that

$$R_n(\mathbf{u} + x) = \sum_{k=0}^{n} R_{n-k}(\mathbf{u})\, x^{\underline{k}}$$

for each $x \in \mathbb{N}$.

[**Hint:** When $\mathbf{u}$ is replaced by $\mathbf{u} + x$, the $\mathbf{u}$-board "grows by x extra (full) columns".]

(b) Now, let $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^n$ be such that $v_1 \geq v_2 \geq \cdots \geq v_n$. Assume further that the n numbers $u_1 + 1, u_2 + 2, \ldots, u_n + n$ are the same as the n numbers $v_1 + 1, v_2 + 2, \ldots, v_n + n$, up to order. (In other words, there exists a permutation $\sigma \in S_n$ such that $u_i + i = v_{\sigma(i)} + \sigma(i)$ for all $i \in [n]$.) Prove that

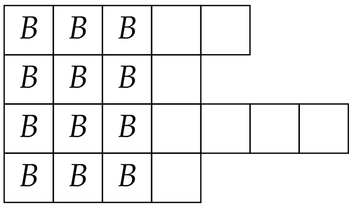
$$R_k(\mathbf{u}) = R_k(\mathbf{v}) \qquad \text{for each } k \in \mathbb{N}.$$

[**Hint:** First, prove that $R_n(\mathbf{u} + x) = R_n(\mathbf{v} + x)$ for each $x \in \mathbb{N}$. Then, use part **(a)** and the previous exercise.]

We notice that Exercise 3 **(b)** is part of [Loehr11, Theorem 12.10]. (The other part is a converse statement: If $R_k(\mathbf{u}) = R_k(\mathbf{v})$ for each $k \in \mathbb{N}$, then the n numbers $u_1 + 1, u_2 + 2, \ldots, u_n + n$ are the same as the n numbers $v_1 + 1, v_2 + 2, \ldots, v_n + n$, up to order.) Our solution to Exercise 3 **(b)** is not a bijective proof; nevertheless, I believe that bijective proofs exist in the literature.

Solution to Exercise 3 (sketched). **(a)** Let $x \in \mathbb{N}$. Consider the "irregular chessboard" $D(\mathbf{u} + x)$. Define a subset B of $\mathbb{Z}^2$ by $B = [n] \times [x]$. Thus, B consists of all the nx squares that lie in rows $1, 2, \ldots, n$ and columns $1, 2, \ldots, x$ of the (infinite) chessboard. All these nx squares belong to $D(\mathbf{u} + x)$ (since all entries of the n-tuple $\mathbf{u} + x$ are $\geq x$, and therefore each row of $D(\mathbf{u} + x)$ has length $\geq x$). In other words, $B \subseteq D(\mathbf{u} + x)$.

For example, if $\mathbf{u} = (2, 1, 4, 1)$ and $x = 3$, then $D(\mathbf{u} + x)$ is the following "irregular chessboard":



(where the cells lying in B are marked with the letter "B").

Let $p \in \mathbb{Z}$.

Each row of $D(\mathbf{u} + x)$ is longer than the corresponding row of $D(\mathbf{u})$ by precisely x cells. If we subtract the set B from $D(\mathbf{u} + x)$, then each row loses its first x cells, and thus it becomes as long as the corresponding row of $D(\mathbf{u})$ but lying x units further right. Hence, the set difference $D(\mathbf{u} + x) \setminus B$ is the same as the set $D(\mathbf{u})$ with all its rows shifted to the right by x units. Hence, there is a bijection

$$\begin{aligned} & \{\text{subsets of } D(\mathbf{u}) \text{ that are rook placements}\} \\ & \to \{\text{subsets of } D(\mathbf{u} + x) \setminus B \text{ that are rook placements}\} \end{aligned}$$

(which simply shifts each rook placement to the right by x units). This bijection clearly preserves the size of a rook placement. Thus, it gives rise (by restriction) to a bijection

$$\begin{aligned} & \{\text{subsets } X \text{ of } D(\mathbf{u}) \text{ that are rook placements and satisfy } |X| = p\} \\ & \to \{\text{subsets } X \text{ of } D(\mathbf{u} + x) \setminus B \text{ that are rook placements and satisfy } |X| = p\}. \end{aligned}$$

Hence,

$$\begin{aligned} & |\{\text{subsets } X \text{ of } D(\mathbf{u}) \text{ that are rook placements and satisfy } |X| = p\}| \\ & = |\{\text{subsets } X \text{ of } D(\mathbf{u} + x) \setminus B \text{ that are rook placements and satisfy } |X| = p\}| \\ & = |\{\text{rook placements in } D(\mathbf{u} + x) \text{ that are disjoint from } B \text{ and satisfy } |X| = p\}| \end{aligned}$$

(because the subsets X of $D(\mathbf{u} + x) \setminus B$ that are rook placements are precisely the rook placements in $D(\mathbf{u} + x)$ that are disjoint from B). Therefore,

$$\begin{aligned} & |\{\text{rook placements in } D(\mathbf{u} + x) \text{ that are disjoint from } B \text{ and satisfy } |X| = p\}| \\ & = |\{\text{subsets } X \text{ of } D(\mathbf{u}) \text{ that are rook placements and satisfy } |X| = p\}| \\ & = |\{\text{rook placements } X \text{ in } D(\mathbf{u}) \text{ that satisfy } |X| = p\}| \\ & \qquad \left(\begin{array}{c}\text{because the subsets of } D(\mathbf{u}) \text{ that are rook placements} \\ \text{are precisely the rook placements in } D(\mathbf{u})\end{array}\right) \\ & = |\{\text{rook placements } X \text{ in } D(\mathbf{u}) \text{ of size } p\}| \\ & = (\text{the number of all rook placements in } D(\mathbf{u}) \text{ of size } p) \\ & = R_p(\mathbf{u}) \qquad (8) \end{aligned}$$

(since $R_p(\mathbf{u})$ was defined to be the number of all rook placements in $D(\mathbf{u})$ of size p).

Now, forget that we fixed p. We thus have proven (8) for each $p \in \mathbb{Z}$.

On the other hand, recall that $R_n(\mathbf{u}+x)$ is the number of all rook placements in $D(\mathbf{u}+x)$ of size n (by the definition of $R_n(\mathbf{u}+x)$). Any such rook placement must contain n rooks, and thus must contain **exactly one rook** in each of the n rows $1,2,\ldots,n$ (this was proven during our proof of Proposition 0.3). Thus, we can construct every rook placement in $D(\mathbf{u}+x)$ of size n by the following algorithm:

- First, we decide how many rooks will lie in B. In other words, we choose the number $k \in \{0,1,\ldots,n\}$ such that k of the n rooks of our rook placement will lie in B.
- Next, we choose the $n-k$ rooks of our rook placement that will not lie in B. These $n-k$ rooks must therefore form a rook placement X in $D(\mathbf{u}+x)$ that is disjoint from B and satisfies $|X| = n-k$. The number of such rook placements is $R_{n-k}(\mathbf{u})$ (by (8), applied to $p = n-k$). Thus, this step can be performed in $R_{n-k}(\mathbf{u})$ many ways.
- Note that X is a rook placement, and thus any two distinct rooks from X occupy distinct rows. Hence, exactly $n-k$ of the n rows $1,2,\ldots,n$ contain a rook from X. We call these $n-k$ rows *occupied*. The remaining k among the n rows $1,2,\ldots,n$ will be called *unoccupied*.
- Next, we choose the remaining k rooks of our rook placement. These k rooks must lie in B, and they must occupy the k unoccupied rows (because no two distinct rooks in a rook placement may occupy the same row). Thus, for each of the k unoccupied rows, we must place a rook in this row in one of the columns $1,2,\ldots,x$ (because this rook must lie in B). This can be done in as follows:
  - First, place a rook in the topmost unoccupied row. This can be done in x many ways (since it must lie in one of the columns $1,2,\ldots,x$).
  - Next, place a rook in the second (from the top) unoccupied row. This can be done in $x-1$ many ways (since it must lie in one of the columns $1,2,\ldots,x$, but not in the same column as the previously placed rook).
  - Next, place a rook in the third (from the top) unoccupied row. This can be done in $x-2$ many ways (since it must lie in one of the columns $1,2,\ldots,x$, but not in any of the same columns as the two previously placed rooks).
  - And so on, until all unoccupied rows have been used up.

  Thus, in total, these last k rooks may be placed in $x(x-1)\cdots(x-k+1)$ many ways.

Thus, the number of rook placements in $D(\mathbf{u}+x)$ of size n is $\sum_{k\in\{0,1,\ldots,n\}} R_{n-k}(\mathbf{u}) \cdot (x(x-1)\cdots(x-k+1))$ (because the above algorithm starts out by choosing $k \in \{0,1,\ldots,n\}$, and then involves $R_{n-k}(\mathbf{u})$ choices in the first step and $x(x-1)\cdots(x-k+1)$ choices in the second). In other words,

$$R_n(\mathbf{u}+x) = \sum_{k\in\{0,1,\ldots,n\}} R_{n-k}(\mathbf{u}) \cdot (x(x-1)\cdots(x-k+1))$$

(since $R_n(\mathbf{u}+x)$ is the number of all rook placements in $D(\mathbf{u}+x)$ of size n). Hence,

$$R_n(\mathbf{u}+x) = \underbrace{\sum_{k\in\{0,1,\ldots,n\}}}_{=\sum_{k=0}^{n}} R_{n-k}(\mathbf{u}) \cdot \underbrace{(x(x-1)\cdots(x-k+1))}_{\substack{=x^{\underline{k}} \\ \text{(by the definition of } x^{\underline{k}})}} = \sum_{k=0}^{n} R_{n-k}(\mathbf{u})\, x^{\underline{k}}.$$

This solves Exercise 3 **(a)**.

(b) Let $x \in \mathbb{N}$. Then, $u_1 + x \geq u_2 + x \geq \cdots \geq u_n + x$ (since $u_1 \geq u_2 \geq \cdots \geq u_n$) and $\mathbf{u}+x = (u_1+x, u_2+x, \ldots, u_n+x)$. Hence, Proposition 0.3 (applied to $\mathbf{u}+x$ and $u_i + x$ instead of $\mathbf{u}$ and u_i) yields

$$\begin{aligned} R_n(\mathbf{u}+x) &= \prod_{i=0}^{n-1} ((u_{n-i}+x) - i) = \prod_{i=1}^{n} \left(\left(\underbrace{u_{n-(n-i)}}_{=u_i} + x \right) - (n-i) \right) \\ &\qquad \text{(here, we have substituted } n-i \text{ for } i \text{ in the product)} \\ &= \prod_{i=1}^{n} \underbrace{((u_i+x) - (n-i))}_{=(u_i+i)-n+x} = \prod_{i=1}^{n} ((u_i+i) - n + x). \end{aligned} \tag{9}$$

Exercise 3 **(a)** yields

$$R_n(\mathbf{u}+x) = \sum_{k=0}^{n} R_{n-k}(\mathbf{u})\, x^{\underline{k}}.$$

Hence,

$$\sum_{k=0}^{n} R_{n-k}(\mathbf{u})\, x^{\underline{k}} = R_n(\mathbf{u}+x) = \prod_{i=1}^{n} ((u_i+i) - n + x) \tag{10}$$

(by (9)). The same argument (applied to $\mathbf{v}$ and v_i instead of $\mathbf{u}$ and u_i) yields

$$\sum_{k=0}^{n} R_{n-k}(\mathbf{v})\, x^{\underline{k}} = \prod_{i=1}^{n} ((v_i+i) - n + x). \tag{11}$$

But we have assumed that the n numbers $u_1+1, u_2+2, \ldots, u_n+n$ are the same as the n numbers $v_1+1, v_2+2, \ldots, v_n+n$, up to order. Hence, the products

$\prod_{i=1}^{n}((u_i+i)-n+x)$ and $\prod_{i=1}^{n}((v_i+i)-n+x)$ contain the same factors, up to order. Therefore, these two products are equal. In other words,

$$\prod_{i=1}^{n}((u_i+i)-n+x)=\prod_{i=1}^{n}((v_i+i)-n+x).$$

Hence, (10) becomes

$$\sum_{k=0}^{n} R_{n-k}(\mathbf{u})\, x^{\underline{k}} = \prod_{i=1}^{n}((u_i+i)-n+x) = \prod_{i=1}^{n}((v_i+i)-n+x) = \sum_{k=0}^{n} R_{n-k}(\mathbf{v})\, x^{\underline{k}} \qquad \text{(by (11))}.$$

Now, forget that we fixed x. We thus have proven that each $x \in \mathbb{N}$ satisfies $\sum_{k=0}^{n} R_{n-k}(\mathbf{u})\, x^{\underline{k}} = \sum_{k=0}^{n} R_{n-k}(\mathbf{v})\, x^{\underline{k}}$. Hence, each $x \in \{0,1,\ldots,n\}$ satisfies $\sum_{k=0}^{n} R_{n-k}(\mathbf{u})\, x^{\underline{k}} = \sum_{k=0}^{n} R_{n-k}(\mathbf{v})\, x^{\underline{k}}$. Thus, Exercise 2 **(b)** (applied to $a_k = R_{n-k}(\mathbf{u})$ and $b_k = R_{n-k}(\mathbf{v})$) yields that $(R_{n-0}(\mathbf{u}), R_{n-1}(\mathbf{u}), \ldots, R_{n-n}(\mathbf{u})) = (R_{n-0}(\mathbf{v}), R_{n-1}(\mathbf{v}), \ldots, R_{n-n}(\mathbf{v}))$. In other words,

$$R_{n-i}(\mathbf{u}) = R_{n-i}(\mathbf{v}) \qquad \text{for each } i \in \{0,1,\ldots,n\}. \tag{12}$$

Recall that we want to prove that $R_k(\mathbf{u}) = R_k(\mathbf{v})$ for each $k \in \mathbb{N}$. So let us fix $k \in \mathbb{N}$. If $k > n$, then $R_k(\mathbf{u}) = R_k(\mathbf{v})$ is obvious[5]. Hence, for the rest of this proof, we WLOG assume that we don't have $k > n$. Thus, $k \leq n$, so that $k \in \{0,1,\ldots,n\}$ and therefore $n-k \in \{0,1,\ldots,n\}$. Applying (12) to $i = n-k$, we thus obtain $R_{n-(n-k)}(\mathbf{u}) = R_{n-(n-k)}(\mathbf{v})$. This simplifies to $R_k(\mathbf{u}) = R_k(\mathbf{v})$. This solves Exercise 3 **(b)**. □

To illustrate the usefulness of Exercise 3, let me express $R_k(2n-1, 2n-3, \ldots, 5, 3, 1)$ in a much simpler way than before:

Corollary 0.4. Let $k \in \mathbb{N}$. Then,

$$R_k(2n-1, 2n-3, \ldots, 5, 3, 1) = \binom{n}{k}^2 k!.$$

Proof of Corollary 0.4. Define an n-tuple $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ by ($u_i = 2(n-i)+1$ for each $i \in [n]$). Thus,

$$\mathbf{u} = (u_1, u_2, \ldots, u_n) = (2n-1, 2n-3, \ldots, 5, 3, 1),$$

so that $u_1 \geq u_2 \geq \cdots \geq u_n$.

[5] *Proof.* Assume that $k > n$. Then, (7) yields $R_k(\mathbf{u}) = 0$. Also, (7) (applied to $\mathbf{v}$ instead of $\mathbf{u}$) yields $R_k(\mathbf{v}) = 0$. Thus, $R_k(\mathbf{u}) = 0 = R_k(\mathbf{v})$. Hence, we have proven $R_k(\mathbf{u}) = R_k(\mathbf{v})$ in the case when $k > n$.

Define an n-tuple $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^n$ by ($v_i = n$ for each $i \in [n]$). Thus,

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) = (n, n, \ldots, n),$$

so that $v_1 \geq v_2 \geq \cdots \geq v_n$.

The n numbers $u_1 + 1, u_2 + 2, \ldots, u_n + n$ are the same as the n numbers $v_1 + 1, v_2 + 2, \ldots, v_n + n$, up to order. (In fact, the former n numbers are $2n, 2n-1, 2n-2, \ldots, n+2, n+1$, whereas the latter n numbers are $n+1, n+2, \ldots, 2n-2, 2n-1, 2n$; these are just two different ways to list all numbers from $n+1$ to $2n$.)

Hence, Exercise 3 **(b)** yields $R_k(\mathbf{u}) = R_k(\mathbf{v})$. In view of $\mathbf{u} = (2n-1, 2n-3, \ldots, 5, 3, 1)$ and $\mathbf{v} = (n, n, \ldots, n)$, this rewrites as $R_k(2n-1, 2n-3, \ldots, 5, 3, 1) = R_k(n, n, \ldots, n)$.

But (6) (applied to $u = n$) yields $R_k(n, n, \ldots, n) = \dbinom{n}{k}\dbinom{n}{k} k!$. Hence,

$$R_k(2n-1, 2n-3, \ldots, 5, 3, 1) = R_k(n, n, \ldots, n) = \binom{n}{k}\binom{n}{k} k! = \binom{n}{k}^2 k!.$$

This proves Corollary 0.4. □

Remark 0.5. Corollary 0.4 has appeared in disguise as Problem 13 on the IMO Shortlist 1997. This problem asked:

> Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. In town A there are n girls and n boys and every girl knows every boy. Let $a(n,k)$ be the number of ways in which k girls can dance with k boys, so that each girl knows her partner. In town B there are n girls and $2n-1$ boys such that girl i knows boys $1, 2, ..., 2i-1$ (and no others). Let $b(n,k)$ be the number of ways in which k girls from town B can dance with k boys from town B so that each girl knows her partner. Show that $a(n,k) = b(n,k)$.

I claim that this follows easily from Corollary 0.4. Namely, define an n-tuple $\mathbf{u} \in \mathbb{N}^n$ by $\mathbf{u} = (2n-1, 2n-3, \ldots, 5, 3, 1)$. Then, each rook placement X in $D(\mathbf{u})$ of size k gives rise to a way in which k girls from town B can dance with k boys from town B so that each girl knows her partner (namely, for each $(i,j) \in X$, we let girl $n+1-i$ dance with boy j). Conversely, every way in which k girls from town B can dance with k boys from town B so that each girl knows her partner gives rise to a rook placement X in $D(\mathbf{u})$ (namely, $X = \{(i,j) \mid \text{girl } n+1-i \text{ dances with boy } j\}$). Thus, we obtain a bijection

$$\begin{aligned} &\{\text{rook placements in } D(\mathbf{u}) \text{ of size } k\} \\ &\to \{\text{ways in which } k \text{ girls from town } B \text{ can dance with } k \text{ boys from town } B \\ &\qquad \text{so that each girl knows her partner}\}. \end{aligned}$$

This shows that the sizes of the two sets involved in this bijection are equal. Since the size of the first set is $R_k(\mathbf{u})$, while the size of the second set is $b(n,k)$, we thus conclude that $R_k(\mathbf{u}) = b(n,k)$.

Similarly, we can prove that $R_k(\mathbf{v}) = a(n,k)$, where $\mathbf{v} \in \mathbb{N}^n$ is the n-tuple defined by $\mathbf{v} = \Bigg(\underbrace{n, n, \ldots, n}_{n \text{ times}}\Bigg)$. But we have $R_k(\mathbf{u}) = R_k(\mathbf{v})$ (as we showed in the proof of Corollary 0.4). Thus, $a(n,k) = R_k(\mathbf{v}) = R_k(\mathbf{u}) = b(n,k)$; this solves the above problem from the IMO Shortlist 1997.

This problem also appears in [AndFen04, Exercise 5.12].

Rook theory is the study of rook placements – not only in boards of the form $D(\mathbf{u})$, but also in more general subsets of $\mathbb{Z}^2$. For example, the permutations of $[n]$ can be regarded as rook placements in the board $[n] \times [n]$, whereas the derangements of $[n]$ can be regarded as rook placements in the board $\{(i,j) \in [n] \times [n] \mid i \neq j\}$ (the "$n \times n$-chessboard without the main diagonal"). The theory has been developed in the 70s at the UMN (by Jay Goldman, J. T. Joichi, Victor Reiner and Dennis White).

0.3. The sum of the "widths" of all inversions of σ

In the following, "number" means "real number" or "complex number" or "rational number", as you prefer (this doesn't make a difference in these exercises).

Exercise 4. Let $n \in \mathbb{N}$. Let $\sigma \in S_n$. Let $a_1, a_2, \ldots, a_n$ be any n numbers. Prove that

$$\sum_{\substack{1 \leq i < j \leq n; \\ \sigma(i) > \sigma(j)}} \left(a_j - a_i\right) = \sum_{i=1}^{n} a_i \left(i - \sigma\left(i\right)\right).$$

[Here, the symbol "$\sum_{\substack{1 \leq i < j \leq n; \\ \sigma(i) > \sigma(j)}}$" means "sum over all pairs $(i,j) \in [n]^2$ satisfying $i < j$ and $\sigma(i) > \sigma(j)$", that is, "sum over all inversions of σ".]

Exercise 4 is [Grinbe16, Exercise 5.23]; we refer to that place for its solution.

0.4. A hollowed-out determinant

In the following, matrices are understood to be matrices whose entries are numbers (see above).

Recall that for any $n \in \mathbb{N}$, the determinant $\det A$ of an $n \times n$-matrix $A = \left(a_{i,j}\right)_{1 \leq i \leq n,\ 1 \leq j \leq n}$ (this notation means that A is an $n \times n$-matrix whose (i,j)-th entry is $a_{i,j}$ for all i and j) is given by

$$\det A = \sum_{\sigma \in S_n} (-1)^{\sigma} \underbrace{a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}}_{= \prod_{i=1}^{n} a_{i,\sigma(i)}} \qquad (13)$$

$$= \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} a_{i,\sigma(i)}. \qquad (14)$$

Here, as usual, S_n denotes the set of all permutations of $[n] = \{1, 2, \ldots, n\}$.

Exercise 5. Let $n \in \mathbb{N}$. Let P and Q be two subsets of $[n]$ such that $|P| + |Q| > n$. Let $A = \left(a_{i,j}\right)_{1 \leq i \leq n,\ 1 \leq j \leq n}$ be an $n \times n$-matrix such that

$$\text{every } i \in P \text{ and } j \in Q \text{ satisfy } a_{i,j} = 0. \qquad (15)$$

Then, prove that $\det A = 0$.

Example 0.6. Applying Exercise 5 to $n = 5$, $P = \{1,3,5\}$ and $Q = \{2,3,4\}$, we see that

$$\det \begin{pmatrix} a_1 & 0 & 0 & 0 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & 0 & 0 & 0 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & 0 & 0 & 0 & e_5 \end{pmatrix} = 0 \qquad \text{for any numbers } a_1, a_5, \ldots, e_5.$$

Exercise 5 is [Grinbe16, Exercise 6.47 **(a)**].

Solution to Exercise 5 (sketched). Let $\sigma \in S_n$.

Assume (for the sake of contradiction) that $\sigma(P) \subseteq [n] \setminus Q$. Thus,

$$\begin{aligned} |\sigma(P)| &\leq |[n] \setminus Q| = \underbrace{|[n]|}_{=n} - |Q| \qquad (\text{since } Q \subseteq [n]) \\ &= n - |Q| < |P| \qquad (\text{since } |P| + |Q| > n). \end{aligned}$$

But the map σ is injective (since it is a permutation); thus, $|\sigma(P)| = |P|$. Hence, $|P| = |\sigma(P)| < |P|$, which is absurd. This contradiction shows that our assumption (that $\sigma(P) \subseteq [n] \setminus Q$) was false. Hence, $\sigma(P) \not\subseteq [n] \setminus Q$. In other words, there exists some $p \in P$ such that $\sigma(p) \notin [n] \setminus Q$. Consider this p.

From $\sigma(p) \in [n]$ and $\sigma(p) \notin [n] \setminus Q$, we obtain $\sigma(p) \in [n] \setminus ([n] \setminus Q) \subseteq Q$. Hence, (15) (applied to $i = p$ and $j = \sigma(p)$) yields $a_{p,\sigma(p)} = 0$.

Now, $a_{p,\sigma(p)}$ is a factor of the product $\prod_{i=1}^{n} a_{i,\sigma(i)}$. Thus, at least one factor of the product $\prod_{i=1}^{n} a_{i,\sigma(i)}$ is 0 (namely, $a_{p,\sigma(p)} = 0$). Hence, the whole product must be 0. In other words, $\prod_{i=1}^{n} a_{i,\sigma(i)} = 0$.

Now, forget that we fixed σ. We thus have shown that $\prod_{i=1}^{n} a_{i,\sigma(i)} = 0$ for each $\sigma \in S_n$. Hence, (14) becomes

$$\det A = \sum_{\sigma \in S_n} (-1)^{\sigma} \underbrace{\prod_{i=1}^{n} a_{i,\sigma(i)}}_{=0} = \sum_{\sigma \in S_n} (-1)^{\sigma} 0 = 0.$$

This solves Exercise 5. □

Remark 0.7. Exercise 5 can also be solved using linear algebra, at least when the entries of our matrix A belong to a field $\mathbb{F}$ (say, $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$). Namely, in this case, we can consider the n-dimensional $\mathbb{F}$-vector space $\mathbb{F}^n$ of all column vectors with n entries. A vector $v = (v_1, v_2, \ldots, v_n)^T$ in $\mathbb{F}^n$ is said to be *P-trivial* if all

$i \in P$ satisfy $v_i = 0$. The set W of all P-trivial vectors $v \in \mathbb{F}^n$ is an $\mathbb{F}$-vector subspace of $\mathbb{F}^n$ of dimension $n - |P|$.

The condition (15) says that for each $j \in Q$, the j-th column of A is P-trivial. Thus, the matrix A has (at least) $|Q|$ many P-trivial columns. Thus, the matrix A has $> n - |P|$ many P-trivial columns (since $|Q| > n - |P|$ (since $|P| + |Q| > n$)). But any collection of $> n - |P|$ many P-trivial columns must necessarily be linearly dependent (because all these columns lie in the $(n - |P|)$-dimensional vector space W). Hence, the matrix A has some linearly dependent columns. Consequently, the matrix A cannot be full-rank; thus, $\det A = 0$.

This argument is a valid solution to Exercise 5 as it is stated. However, the combinatorial solution given before is "better" because it immediately generalizes to situations where the entries of A don't belong to a field but merely belong to a commutative ring. (The solution using linear algebra can also be extended to these situations, but this requires more work, since much of linear algebra only works over a field.)

0.5. Arrowhead matrices

Exercise 6. Let n be a positive integer.

(a) A permutation $\sigma \in S_n$ will be called *arrowheaded* if each $i \in [n-1]$ satisfies $\sigma(i) = i$ or $\sigma(i) = n$.

[For example, the permutation in S_5 whose one-line notation is $[1, 5, 3, 4, 2]$ is arrowheaded.]

Describe all arrowheaded permutations $\sigma \in S_n$ and find their number.

(b) Given n numbers $a_1, a_2, \ldots, a_n$ as well as $n-1$ numbers $b_1, b_2, \ldots, b_{n-1}$ and $n-1$ further numbers $c_1, c_2, \ldots, c_{n-1}$. Let A be the $n \times n$-matrix

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 & c_1 \\ 0 & a_2 & \cdots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & c_{n-1} \\ b_1 & b_2 & \cdots & b_{n-1} & a_n \end{pmatrix}.$$

(This is the matrix whose (i, j)-th entry is $\begin{cases} a_i, & \text{if } i = j; \\ b_j, & \text{if } i = n \text{ and } j \neq n; \\ c_i, & \text{if } i \neq n \text{ and } j = n; \\ 0, & \text{if } i \neq n \text{ and } j \neq n \text{ and } i \neq j \end{cases}$ for all $i \in [n]$ and $j \in [n]$.) Prove that

$$\det A = a_1 a_2 \cdots a_n - \sum_{i=1}^{n-1} b_i c_i \prod_{\substack{j \in [n-1]; \\ j \neq i}} a_j.$$

The matrix A is called a "reverse arrowhead matrix" due to the shape that its nonzero entries form. ("Reverse" because the usual arrowhead matrix has its arrow pointing northwest rather than southeast.)

Exercise 6 **(b)** is a fairly easy exercise on evaluating determinants, and can be solved in various ways, some of which use no combinatorics at all. Notice also that Exercise 6 **(b)** is equivalent to [Grinbe16, Exercise 6.60 **(b)**] (applied to $n-1$, a_i, b_i and c_i instead of n, d_i, v_i and u_i). We shall give a purely combinatorial solution to Exercise 6, which uses the following lemma:

Lemma 0.8. Let $n \in \mathbb{N}$ and $p \in [n]$. Let $\alpha \in S_n$ and $\beta \in S_n$. Assume that

$$\alpha(i) = \beta(i) \qquad \text{for each } i \in [n] \setminus \{p\}.$$

Then, $\alpha = \beta$.

Lemma 0.8 is [Grinbe16, Lemma 7.232], but you're likely to prove it faster than you can look it up in [Grinbe16]. (Also, I might have mentioned it in class.) Basically, Lemma 0.8 says that a permutation in S_n is uniquely determined by any $n-1$ of its values.

Solution to Exercise 6 (sketched). Recall that if u and v are two distinct elements of $[n]$, then $t_{u,v}$ denotes the permutation in S_n that swaps u with v and leaves all other numbers unchanged. This is called the transposition of u with v.

(a) We claim the following:

Observation 1: The arrowheaded permutations $\sigma \in S_n$ are precisely the $n-1$ transpositions $t_{1,n}, t_{2,n}, \ldots, t_{n-1,n}$ and the identity map id.

[*Proof of Observation 1:* It is trivial to check that the $n-1$ transpositions $t_{1,n}, t_{2,n}, \ldots, t_{n-1,n}$ and the identity map id are arrowheaded. It thus remains to prove that these are the **only** arrowheaded permutations $\sigma \in S_n$. In other words, it remains to prove that every arrowheaded permutation $\sigma \in S_n$ is one of these $n-1$ transpositions $t_{1,n}, t_{2,n}, \ldots, t_{n-1,n}$ or is the identity map id.

Assume the contrary. Thus, there exists some arrowheaded permutation $\sigma \in S_n$ that is neither one of the $n-1$ transpositions $t_{1,n}, t_{2,n}, \ldots, t_{n-1,n}$ nor is the identity map id. Consider this σ. There must exist some $j \in [n-1]$ such that $\sigma(j) \neq j$ [6]. Consider such a j. Since σ is arrowheaded, we know that each $i \in [n-1]$ satisfies $\sigma(i) = i$ or $\sigma(i) = n$. Applying this to $i = j$, we obtain $\sigma(j) = j$ or $\sigma(j) = n$. Since $\sigma(j) \neq j$, we thus have $\sigma(j) = n = t_{j,n}(j)$ (since the transposition $t_{j,n}$ maps j to n).

Now, let $i \in [n-1]$ be arbitrary. We shall show that $\sigma(i) = t_{j,n}(i)$.

[6] *Proof.* Assume the contrary. Thus, $\sigma(i) = i$ for each $i \in [n-1]$. In other words, $\sigma(i) = i$ for each $i \in [n] \setminus \{n\}$ (since $[n-1] = [n] \setminus \{n\}$). Hence, $\sigma(i) = i = \operatorname{id}(i)$ for each $i \in [n] \setminus \{n\}$. Hence, Lemma 0.8 (applied to $p = n$, $\alpha = \sigma$ and $\beta = \operatorname{id}$) yields $\sigma = \operatorname{id}$. This contradicts $\sigma \neq \operatorname{id}$. This contradiction proves that our assumption was false, qed.

Indeed, if $i = j$, then this follows immediately from $\sigma(j) = t_{j,n}(j)$. Hence, for the rest of this proof, we WLOG assume that $i \neq j$. Hence, $\sigma(i) \neq \sigma(j)$ (since σ is injective (since σ is a permutation)). Thus, $\sigma(i) \neq \sigma(j) = n$.

Since $i \neq j$ and $i \neq n$ (because $i \in [n-1]$), we have $t_{j,n}(i) = i$. But recall that $i \in [n-1]$; thus, $\sigma(i) = i$ or $\sigma(i) = n$ (since σ is arrowheaded). Since $\sigma(i) \neq n$, we thus conclude that $\sigma(i) = i = t_{j,n}(i)$. Thus, we have proven that $\sigma(i) = t_{j,n}(i)$.

Now, forget that we fixed i. We thus have shown that $\sigma(i) = t_{j,n}(i)$ for each $i \in [n-1]$. In other words, $\sigma(i) = t_{j,n}(i)$ for each $i \in [n] \setminus \{n\}$ (since $[n-1] = [n] \setminus \{n\}$). Hence, Lemma 0.8 (applied to $p = n$, $\alpha = \sigma$ and $\beta = t_{j,n}$) yields $\sigma = t_{j,n}$. This contradicts the fact that σ is none of the $n-1$ transpositions $t_{1,n}, t_{2,n}, \ldots, t_{n-1,n}$. This contradiction concludes our proof of Observation 1.]

Observation 1 describes all arrowheaded permutations $\sigma \in S_n$. We thus only need to find their number.

The $n-1$ transpositions $t_{1,n}, t_{2,n}, \ldots, t_{n-1,n}$ and the identity map id are n distinct permutations (indeed, they are distinct, since their values at n are distinct). Thus, Observation 1 yields that the number of all arrowheaded permutations $\sigma \in S_n$ is n. This completes the solution of Exercise 6 **(a)**.

(b) Write the $n \times n$-matrix A in the form $A = \left(a_{i,j}\right)_{1 \leq i \leq n,\ 1 \leq j \leq n}$. Thus, for each $i \in [n]$ and $j \in [n]$, we have

$$a_{i,j} = \begin{cases} a_i, & \text{if } i = j; \\ b_j, & \text{if } i = n \text{ and } j \neq n; \\ c_i, & \text{if } i \neq n \text{ and } j = n; \\ 0, & \text{if } i \neq n \text{ and } j \neq n \text{ and } i \neq j \end{cases}.$$

Hence, in particular,

$$a_{i,j} = 0 \qquad \text{whenever } i \neq n \text{ and } j \neq n \text{ and } i \neq j. \tag{16}$$

Hence, if a permutation $\sigma \in S_n$ is not arrowheaded, then

$$\prod_{i=1}^{n} a_{i,\sigma(i)} = 0. \tag{17}$$

[*Proof of (17):* Let $\sigma \in S_n$ be a permutation that is not arrowheaded. Thus, not every $i \in [n-1]$ satisfies $\sigma(i) = i$ or $\sigma(i) = n$ (by the definition of "arrowheaded"). In other words, there exists some $p \in [n-1]$ satisfying $\sigma(p) \neq p$ and $\sigma(p) \neq n$. Consider this p. We have $p \neq n$ (since $p \in [n-1]$) and $\sigma(p) \neq n$ and $p \neq \sigma(p)$. Hence, (16) (applied to $i = p$ and $j = \sigma(p)$) yields $a_{p,\sigma(p)} = 0$. Thus, one of the factors of the product $\prod_{i=1}^{n} a_{i,\sigma(i)}$ is zero (namely, the factor $a_{p,\sigma(p)} = 0$). Hence, the whole product is zero. This proves (17).]

Now, (14) yields

$$\begin{aligned}\det A &= \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^n a_{i,\sigma(i)} \\ &= \sum_{\substack{\sigma\in S_n;\\ \sigma \text{ is arrowheaded}}} (-1)^\sigma \prod_{i=1}^n a_{i,\sigma(i)} + \sum_{\substack{\sigma\in S_n;\\ \sigma \text{ is not arrowheaded}}} (-1)^\sigma \underbrace{\prod_{i=1}^n a_{i,\sigma(i)}}_{\substack{=0\\ \text{(by (17))}}} \\ &= \sum_{\substack{\sigma\in S_n;\\ \sigma \text{ is arrowheaded}}} (-1)^\sigma \prod_{i=1}^n a_{i,\sigma(i)}. \qquad (18)\end{aligned}$$

But Observation 1 above shows that the arrowheaded permutations $\sigma \in S_n$ are precisely the $n-1$ transpositions $t_{1,n}, t_{2,n}, \ldots, t_{n-1,n}$ and the identity map id. Moreover, as we have seen in our solution of Exercise 6 **(a)**, these are altogether n distinct permutations. Hence, the sum on the right hand side of (18) can be rewritten as follows:

$$\begin{aligned}&\sum_{\substack{\sigma\in S_n;\\ \sigma \text{ is arrowheaded}}} (-1)^\sigma \prod_{i=1}^n a_{i,\sigma(i)} \\ &= \sum_{k=1}^{n-1} \underbrace{(-1)^{t_{k,n}}}_{\substack{=-1\\ \text{(since } t_{k,n} \text{ is a transposition)}}} \prod_{i=1}^n a_{i,t_{k,n}(i)} + \underbrace{(-1)^{\operatorname{id}}}_{=1} \prod_{i=1}^n \underbrace{a_{i,\operatorname{id}(i)}}_{\substack{=a_{i,i}=a_i\\ \text{(by the definition of } A)}} \\ &= \sum_{k=1}^{n-1} (-1) \prod_{i=1}^n a_{i,t_{k,n}(i)} + \prod_{i=1}^n a_i. \qquad (19)\end{aligned}$$

But each $k \in [n-1]$ satisfies

$$\begin{aligned}\prod_{i=1}^n a_{i,t_{k,n}(i)} &= \underbrace{a_{n,t_{k,n}(n)}}_{\substack{=a_{n,k}\\ \text{(since } t_{k,n}(n)=k)}} \underbrace{a_{k,t_{k,n}(k)}}_{\substack{=a_{k,n}\\ \text{(since } t_{k,n}(k)=n)}} \prod_{\substack{i\in[n];\\ i\neq n \text{ and } i\neq k}} \underbrace{a_{i,t_{k,n}(i)}}_{\substack{=a_{i,i}\\ \text{(since } t_{k,n}(i)=i\\ \text{(because } i\neq k \text{ and } i\neq n))}} \\ &\qquad \left(\begin{array}{c}\text{here, we have split off the factors for } i=n \text{ and for } i=k\\ \text{from the product}\end{array}\right) \\ &= \underbrace{a_{n,k}}_{\substack{=b_k\\ \text{(by the definition of } A)}} \underbrace{a_{k,n}}_{\substack{=c_k\\ \text{(by the definition of } A)}} \underbrace{\prod_{\substack{i\in[n];\\ i\neq n \text{ and } i\neq k}}}_{=\prod_{\substack{i\in[n-1];\\ i\neq k}}} \underbrace{a_{i,i}}_{\substack{=a_i\\ \text{(by the definition of } A)}} \\ &= b_k c_k \prod_{\substack{i\in[n-1];\\ i\neq k}} a_i = b_k c_k \prod_{\substack{j\in[n-1];\\ j\neq k}} a_j\end{aligned}$$

(here, we have renamed the index i as j in the product). Hence, (19) becomes

$$\begin{aligned}
&\sum_{\substack{\sigma\in S_n;\\ \sigma\text{ is arrowheaded}}} (-1)^{\sigma} \prod_{i=1}^{n} a_{i,\sigma(i)} \\
&= \sum_{k=1}^{n-1} (-1) \underbrace{\prod_{i=1}^{n} a_{i,t_{k,n}(i)}}_{=b_k c_k \prod_{\substack{j\in[n-1];\\ j\neq k}} a_j} + \prod_{i=1}^{n} a_i \\
&= \underbrace{\sum_{k=1}^{n-1} (-1)\, b_k c_k \prod_{\substack{j\in[n-1];\\ j\neq k}} a_j}_{=-\sum_{k=1}^{n-1} b_k c_k \prod_{\substack{j\in[n-1];\\ j\neq k}} a_j} + \underbrace{\prod_{i=1}^{n} a_i}_{=a_1a_2\cdots a_n} \\
&= -\sum_{k=1}^{n-1} b_k c_k \prod_{\substack{j\in[n-1];\\ j\neq k}} a_j + a_1a_2\cdots a_n = a_1a_2\cdots a_n - \sum_{k=1}^{n-1} b_k c_k \prod_{\substack{j\in[n-1];\\ j\neq k}} a_j \\
&= a_1a_2\cdots a_n - \sum_{i=1}^{n-1} b_i c_i \prod_{\substack{j\in[n-1];\\ j\neq i}} a_j
\end{aligned}$$

(here, we have renamed the summation index k as i). Thus, (18) becomes

$$\det A = \sum_{\substack{\sigma\in S_n;\\ \sigma\text{ is arrowheaded}}} (-1)^{\sigma} \prod_{i=1}^{n} a_{i,\sigma(i)} = a_1a_2\cdots a_n - \sum_{i=1}^{n-1} b_i c_i \prod_{\substack{j\in[n-1];\\ j\neq i}} a_j.$$

This solves Exercise 6 **(b)**. □

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The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .

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