3.7. Compositions, weak compositions, multisets

How many ways are there to write 5 as a sum of 3 positive integers, if the order matters? 6, namely:

$$5 = 1+2+2 = 2+1+2 = 2+2+1$$
$$= 1+1+3 = 1+3+1 = 3+1+1.$$

Theorem 3.23. Let $n, k \in \mathbb{N}$. Let $\mathbb{P} = \{1, 2, 3, \ldots\}$.
Then, (# of $(x_1, \ldots, x_k) \in \mathbb{P}^k$ satisfying $x_1 + \cdots + x_k = n$)

$$= \begin{cases} \binom{n-1}{k-1} & \text{if } n \geq 1; \\ [k=0] & \text{if } n = 0. \end{cases}$$

Remark. ~~The k-tuples~~ A <u>composition</u> is a tuple of positive integers. A <u>composition into k parts</u> is a k-tuple of positive integers. A <u>composition of n</u> is a tuple of positive integers whose sum is n. So, Thm. 3.23 says:

$$(\text{# of compositions of } n \text{ into } k \text{ parts}) = \begin{cases} \binom{n-1}{k-1} & \text{if } n \geq 1; \\ [k=0] & \text{if } n = 0. \end{cases}$$

Proof. WLOG assume $n \geq 1$ (since the $n = 0$ case is trivial).

Consider, for each k-tuple $(x_1, \ldots, x_k) \in \mathbb{P}^k$ satisfying $x_1 + \cdots + x_k = n$, the set

$$\begin{aligned} D(x_1, \ldots, x_k) &= \{ x_1 + \cdots + x_j \mid 1 \leq j < k \} \\ &= \{ x_1, \\ &\qquad x_1 + x_2, \\ &\qquad \vdots \\ &\qquad x_1 + \cdots + x_{k-1} \}; \end{aligned}$$

~~The map~~ it is a $(k-1)$-element subset of $[n-1]$

The map

$$\{k\text{-tuples } (x_1, \ldots, x_k) \in \mathbb{P}^k \text{ satisfying } x_1 + \cdots + x_k = n\}$$
$$\to \{(k-1)\text{-element subsets of } [n-1]\},$$
$$(x_1, \ldots, x_k) \mapsto D(x_1, \ldots, x_k)$$

is a bijection (indeed, the inverse map sends a $(k-1)$-element subset $\{s_1 < s_2 < \cdots < s_k\}$ of $[n-1]$ to

$(s_1 - s_0, s_2 - s_1, s_3 - s_2, \ldots, s_{k+1} - s_k)$, where we set $s_0 = 0$ and $s_{k+1} = n$).

Thus,

$$(\# \text{ of } k\text{-tuples } (x_1, \ldots, x_k) \in P^k \text{ satisfying } x_1 + \cdots + x_k = n)$$
$$= (\# \text{ of } (k-1)\text{-element subsets of } [n-1])$$
$$= \binom{n-1}{k-1} \quad (\text{since } n \geq 1),$$

$\square$

qed.

Theorem 3.24. Let $n, k \in \mathbb{N}$. Then,

$$(\# \text{ of } (x_1, \ldots, x_k) \in \{0,1\}^k \text{ satisfying } x_1 + \cdots + x_k = n)$$
$$= \binom{k}{n}.$$

Proof. To construct such a tuple, we just need to decide which k i's will satisfy $x_i = 1$. There are $\binom{k}{n}$ choices for this.

Rigorous version: The map

$$\{\text{subsets of } [k]\} \longrightarrow \{0,1\}^k,$$

$$S \longmapsto \left([1 \in S], [2 \in S], \ldots, [k \in S] \right)$$

is a bijection. Thus,

$$\begin{aligned}
&\left(\# \text{ of } (x_1, \ldots, x_k) \in \{0,1\}^k \text{ satisfying } x_1 + \cdots + x_k = n \right) \\
&= \Big(\# \text{ of subsets } S \text{ of } [k] \text{ satisfying } \underbrace{[1 \in S] + \cdots + [k \in S]}_{\substack{= |S| \\ \text{(by Lemma 3.24.2} \\ \text{below)}}} = n \Big) \\
&= \left(\# \text{ of subsets } S \text{ of } [k] \text{ satisfying } |S| = n \right) = \binom{k}{n}. \qquad \square
\end{aligned}$$

Lemma 3.24.2. ~~Let G be a set. Let S be~~ ("Counting by roll-call"). Let G be a set. Let S be a finite subset of G. Then,

$$|S| = \sum_{g \in G} [g \in S].$$

Proof. Easy, but the lemma is highly useful.

Theorem 3.25. Let $n, k \in \mathbb{N}$. Then,

$$(\# \text{ of } (x_1, \dots, x_k) \in \mathbb{N}^k \text{ satisfying } x_1 + \dots + x_k = n)$$

$$= \begin{cases} \binom{n+k-1}{k-1} & \text{if } k \neq 0; \\ [n=0] & \text{if } k = 0 \end{cases}$$

$$= \binom{n+k-1}{n}.$$

Remark. ~~The~~ Tuples of nonnegative integers are called **weak compositions.**

Example: For $n=2$ and $k=3$, the # of weak compositions of 2 into 3 parts is $\binom{2+3-1}{2} = 6$; and they are

$$2 = 0+1+1 = 1+0+1 = 1+1+0$$
$$= 0+0+2 = 0+2+0 = 2+0+0.$$

Proof of Theorem 3.25. The cases $k=0$ and $n=0$ are again omitted. So $k>0$ and $n>0$. Let $\mathbb{P} = \{1, 2, 3, \dots\}$.

The map

$$\{(x_1,\dots,x_k) \in \mathbb{N}^k \mid x_1+\dots+x_k = n\} \to \{(x_1,\dots,x_k) \in \mathbb{P}^k \mid x_1+\dots+x_k = n+k\},$$

$$(x_1,\dots,x_k) \longmapsto (x_1+1,\dots,x_k+1)$$

is a bijection. Thus,

$$(\# \text{ of } (x_1,\dots,x_k) \in \mathbb{N}^k \text{ satisfying } x_1+\dots+x_k = n)$$

$$= (\# \text{ of } (x_1,\dots,x_k) \in \mathbb{P}^k \text{ satisfying } x_1+\dots+x_k = n+k)$$

$$= \begin{cases} \binom{n+k-1}{k-1} & \text{if } n+k \geq 1 \\ [k=0] & \text{if } n+k=0 \end{cases} \quad \left(\text{by Theorem 3.23, applied to } n+k \text{ instead of } n\right)$$

$$= \binom{n+k-1}{k-1} = \binom{n+k-1}{n} \quad (\text{by symmetry}). \qquad \square$$

Theorem 3.25 is related to multisets.

Def. Let S be a set.

A ~~finite~~ multiset is, roughly speaking, a "set" whose elements can appear with multiplicities.

All we care about here are <u>finite multisubsets of S</u>, which ~~are~~ can be defined as maps $S \to \mathbb{N}$. (The value of such a map at $t \in S$ is the "multiplicity" of t in the multisubset).

For example, "the multisubset $\{1, 4, 4, 5, 7, 7, 7\}$ of $[8]$" is encoded as the map $[8] \to \mathbb{N}$ sending $1 \mapsto 1$, $2 \mapsto 0$, $3 \mapsto 0$, $4 \mapsto 2$, $5 \mapsto 1$, $6 \mapsto 0$, $7 \mapsto 3$, $8 \mapsto 0$. The <u>size</u> of a multisubset f is defined as $\sum_{t \in S} f(t)$.

<u>Corollary 3.26.</u> Let $n, k \in \mathbb{N}$. Let S be an k-elt. set. Then, the # of multisubsets of S having size n is $\binom{n+k-1}{n}$.

<u>Proof.</u> Let $s_1, \ldots, s_k$ be the k elements of S. Then,

$$\{\text{multisubsets of } S \text{ having size } n\} \to \{(x_1, \ldots, x_k) \in \mathbb{N}^k \mid x_1 + \cdots + x_k = n\},$$

$$f \mapsto (f(s_1), \ldots, f(s_k))$$

is a bijection.

Now, use Theorem 3.25. $\square$

<u>Remark.</u> HW2 exercise 3(a) asked you to show that

if $m, a, b \in \mathbb{N}$, then;

(# of lacunar subsets of $[2m]$ with exactly a even & b odd elements)

(1) $= [a \leq m][b \leq m]\binom{m-a}{b}\binom{m-b}{a}$.

We can solve this using multisets;

Let S be a lacunar subset of $[2m]$ with exactly a even & b odd elements.

Write S as $\{s_1 < s_2 < \dots < s_{a+b}\}$.

Then, $s_1 - 0 \leq s_2 - 2 \leq s_3 - 4 \leq \dots \leq s_{a+b} - 2(a+b-1)$.

Now, consider the multisubset

$$M := \{s_1 - 0, s_2 - 2, s_3 - 4, \dots, s_{a+b} - 2(a+b-1)\}_{\text{multiset}}$$

of $[2m - 2(a+b-1)]$. It has exactly a even & b odd elements. So we can split M into 2 "multiset union" $M_{\text{even}} \dot{\cup} M_{\text{odd}}$ (like a union of sets, but multiplicities get added), where

M_{even} is a size-a multisubset of
$\{2, 4, 6, \ldots, 2m - 2(a+b-1)\}$,

and where

M_{odd} is a size-b multisubset of
$\{1, 3, 5, \ldots, 2m - 2(a+b-1) - 1\}$.

Moreover, this encoding ~~is~~

$\{$lacunar subsets of $[2m]$ with exactly a even & b odd elements$\}$

$\to \{$size-a multisubsets of $\{2, 4, 6, \ldots, 2m - 2(a+b-1)\}\}$
$\times \{$size-b multisubsets of $\{1, 3, 5, \ldots, 2m - 2(a+b-1) - 1\}\}$,

$S \mapsto (M_{even}, M_{odd})$

is a bijection. Therefore, the LHS of (1) is

$$\underbrace{|\{\text{size-}a \text{ multisubsets of } \{2,4,6,\ldots, 2m-2(a+b-1)\}\}|}_{= \binom{m-(a+b-1)+a-1}{a} \text{ (by Cor. 3.26)}}$$

$= [\text{something}] \binom{m-(a+b-1)+a-1}{a}$

$$\underbrace{\left|\{\text{size-}b \text{ multisubsets of } \{1,3,5,\ldots,2m-2(a+b-1)-1\}\}\right|}_{=[\text{something}]\binom{m-(a+b-1)+b-1}{b}} \quad (\text{by Cor. } 3.26)$$

$$= \underbrace{[\text{something}]\cdot[\text{something}]}_{=(1 \text{ if } a\leq m \ \& \ b\leq m)}\cdot \underbrace{\binom{m-(a+b-1)+a-1}{a}}_{=\binom{m-b}{a}}\underbrace{\binom{m-(a+b-1)+b-1}{b}}_{=\binom{m-a}{b}}$$

$$= \binom{m-b}{a}\binom{m-a}{b} \quad \text{if } a\leq m \ \& \ b\leq m,$$

If $a>m$, then no such subsets exist $\Rightarrow 0=0$. $\square$

-//- $b>m$

Similarly, we can do Exercise 3 (b).

3.~~4~~8. Destructive interference and the principle of inclusion & exclusion.

Theorem 3.27. (Principle of Inclusion & Exclusion, aka the Sylvester sieve formula). Let $n \in \mathbb{N}$. Let $A_1, A_2, \ldots, A_n$ be finite sets.

(a) We have

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| = {} & |A_1| + |A_2| + \cdots + |A_n| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n| \\ & + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \cdots \\ & - \cdots \\ & + \cdots \\ & \pm \cdots \\ & + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|.\end{aligned}$$

Or, in rigorous terms:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\substack{I \subseteq [n];\\ I \neq \varnothing}} (-1)^{|I|-1} \left|\bigcap_{i \in I} A_i\right|.$$

(Here, $\bigcap_{i \in I} A_i$ means "the intersection of all A_i with $i \in I$".
E.g., $\bigcap_{i \in \{2,3,6\}} A_i = A_2 \cap A_3 \cap A_6$.)

(b) Let U be a finite set that contains all A_i's as subsets. Then

$$\left|U \setminus \bigcup_{i=1}^{n} A_i\right| = \sum_{I \subseteq [n]} (-1)^{|I|} \left|\bigcap_{i \in I} A_i\right|,$$

where we set $\bigcap_{i \in \varnothing} A_i = U$.

Vic has sketched an induction proof. See [Galvin, §16] for 2 other proofs. Here, we'll show a different proof, based on the following theorem:

Thm. 3.28 ("Destructive interference").

Let G be a finite set. Then,

$$\sum_{I \subseteq G} (-1)^{|I|} = [G = \varnothing].$$

Example: If $G = \{1,2,3\}$, then

$$\sum_{I \subseteq G} (-1)^{|I|} = \cancel{(-1)^{|\varnothing|}} + \cancel{(-1)^{|\{1\}|}} + (-1)^{|\{2\}|} + (-1)^{|\{1,2\}|}$$
$$+ \cancel{(-1)^{|\{3\}|}} + \cancel{(-1)^{|\{1,3\}|}} + (-1)^{|\{2,3\}|} + (-1)^{|\{1,2,3\}|}$$
$$= 1 + (-1) + (-1) + 1$$
$$+ (-1) + 1 + 1 + (-1)$$
$$= 0 = [\{1,2,3\} = \varnothing].$$

Proof. Idea: ~~Each I that doesn't cont~~ If $G = \varnothing$, then this is obvious. So WLOG assume that G is nonempty. Fix any $g \in G$. Then, the $I \subseteq G$ that contain g have their addends cancel the addends corresponding to the I that don't contain g. $\square$

Proof of Theorem 3.27.

(b) Let $x \in U$. Let $G = \{ i \in [n] \mid x \in A_i \} \subseteq [n]$

Now,

$$\sum_{I \subseteq [n]} (-1)^{|I|} \underbrace{\left[x \in \bigcap_{i \in I} A_i \right]}_{\substack{= [x \in A_i \text{ for all } i \in I] \\ = [i \in G \text{ for all } i \in I] \\ = [I \subseteq G]}}$$

$$= \sum_{I \subseteq [n]} (-1)^{|I|} \cdot [I \subseteq G]$$

$$= \sum_{\substack{I \subseteq [n]; \\ I \subseteq G}} (-1)^{|I|} \cdot \underbrace{[I \subseteq G]}_{=1} + \sum_{\substack{I \subseteq [n]; \\ I \not\subseteq G}} (-1)^{|I|} \cdot \underbrace{[I \subseteq G]}_{\substack{=0 \\ (\text{since } I \not\subseteq G)}}$$

$$= \sum_{\substack{I \subseteq [n]; \\ I \subseteq G}} (-1)^{|I|} = \sum_{I \subseteq G} (-1)^{|I|} = [G = \varnothing]$$

$= [\text{there exist no } i \in [n] \text{ such that } x \in A_i]$

$= [x \notin A_i \text{ for all } i \in [n]]$

(2) $= [x \notin \bigcup_{i=1}^{n} A_i] = [x \in U \setminus \bigcup_{i=1}^{n} A_i]$.

This holds $\forall x \in U$.

Now,

$$\sum_{I \subseteq [n]} (-1)^{|I|} \underbrace{\left| \bigcap_{i \in I} A_i \right|}_{= \sum_{x \in U} [x \in \bigcap_{i \in I} A_i]}$$

(by Lemma 3.242)

$$= \sum_{I \subseteq [n]} (-1)^{|I|} \sum_{x \in U} [x \in \bigcap_{i \in I} A_i]$$

$$= \sum_{x \in U} \underbrace{\sum_{I \subseteq [n]} (-1)^{|I|} [x \in \bigcap_{i \in I} A_i]}_{\overset{(2)}{=} [x \in U \setminus \bigcup_{i=1}^{n} A_i]}$$

$$= \sum_{x \in U} \left[x \in U \setminus \bigcup_{i=1}^{n} A_i \right]$$

$$= \left| U \setminus \bigcup_{i=1}^{n} A_i \right| \qquad \text{(by Lemma 3.24 a).}$$

This proves Theorem 3.27 (b).

(a). ~~Clearly, Fix~~ Let $U = \bigcup_{i=1}^{n} A_i$. Then,

$\left| U \setminus \bigcup_{i=1}^{n} A_i \right| = 0$. But (b) yields

$$\left| U \setminus \bigcup_{i=1}^{n} A_i \right| = \sum_{I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|$$

$$= \underbrace{(-1)^{|\varnothing|}}_{=1} \left| \underbrace{\bigcap_{i \in \varnothing} A_i}_{=U} \right| + \sum_{\substack{I \subseteq [n]; \\ I \neq \varnothing}} \underbrace{(-1)^{|I|}}_{= -(-1)^{|I|-1}} \left| \bigcap_{i \in I} A_i \right|$$

$$= |U| - \sum_{\substack{I \subseteq [n]; \\ I \neq \varnothing}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|.$$

Comparing these, we get

$$|U| - \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right| = 0,$$

so that
$$|U| = \sum_{\substack{I \subseteq [n]; \\ I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|.$$

Since $U = \bigcup_{i=1}^{n} A_i$, this is exactly Theorem 3.27(a). $\square$

Rmk. If n is a positive integer, then $\phi(n)$ denotes the number of all $i \in [n]$ that are coprime (= relatively prime) to n.

For example, $\phi(6) = |\{1, \cancel{2}, \cancel{3}, \cancel{4}, 5, \cancel{6}\}| = 2$.

There is a formula for ϕ: If $p_1, p_2, \ldots, p_k$ are the distinct primes dividing n, then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$

See [LeLeMe, §15.9.5] for a proof using Theorem 3.27.