<u>Recall:</u> An <u>inversion</u> of a permutation $\sigma \in S_n$ is a pair (i,j) satisfying $1 \le i < j \le n$ and $\sigma(i) > \sigma(j)$.

The <u>length</u> $\ell(\sigma)$ of a permutation $\sigma \in S_n$ is the # of inversions of σ.

<u>Example:</u> If $\sigma = [3,1,5,2,4] \in S_5$ (in one-line notation), then

- the inversions of σ are $(1,2), (1,4), (3,4), (3,5)$;
- the length of σ is 4.

Furthermore, for this σ, we have $\sigma \circ s_1 = [1,3,5,2,4]$, and

- the inversions of $\sigma \circ s_1$ are $(2,4), (3,4), (3,5)$;
- the length of $\sigma \circ s_1$ is 3.

Also, for this σ, we have $\sigma \circ s_2 = [3,5,1,2,4]$, and

- the inversions of $\sigma \circ s_2$ are $(1,3), (1,4), (2,3), (2,4), (2,5)$;
- the length of $\sigma \circ s_2$ is 5.

General rule: The one-line notation for $\sigma \circ s_k$ ~~obtain~~ (where $k \in [n-1]$) is obtained from the one-line notation for σ by swapping the k-th and $(k+1)$-st entries. The effect on inversions is:

- If $\sigma(k) < \sigma(k+1)$, then $\sigma \circ s_k$ has a new inversion $(k, k+1)$.
  If $\sigma(k) > \sigma(k+1)$, then σ had an inversion $(k, k+1)$, which $\sigma \circ s_k$ no longer has.
- Any inversion (i,j) of σ with $(i,j) \neq (k,k+1)$ gives rise to an inversion $(s_k(i), s_k(j))$ of $\sigma \circ s_k$.
- This covers all inversions of $\sigma \circ s_k$.

$$\Longrightarrow \quad (1) \begin{cases} \ell(\sigma \circ s_k) = \ell(\sigma) + 1 & \text{if } \sigma(k) < \sigma(k+1); \\ \ell(\sigma \circ s_k) = \ell(\sigma) - 1 & \text{if } \sigma(k) > \sigma(k+1). \end{cases}$$

<u>Prop. 5.4.</u> For any $\sigma \in S_n$, we have $\ell(\sigma^{-1}) = \ell(\sigma)$.

<u>Proof.</u> There is a bijection

$$\begin{aligned} \{\text{inversions of } \sigma\} &\longrightarrow \{\text{inversions of } \sigma^{-1}\}, \\ (i,j) &\longmapsto (\sigma^{-1}(j), \sigma^{-1}(i)). \end{aligned}$$

(Its inverse map sends (u,v) to $(\sigma(v), \sigma(u))$.)

For details: [detnotes, Exercise 4.2 (f)]. $\square$

Prop. 5.5. Let $n \in \mathbb{N}$, $\sigma \in S_n$ and $k \in [n-1]$.

(a) We have

$$\ell(\sigma \circ s_k) = \begin{cases} \ell(\sigma)+1, & \text{if } \sigma(k) < \sigma(k+1); \\ \ell(\sigma)-1, & \text{if } \sigma(k) > \sigma(k+1). \end{cases}$$

(b) We have

$$\ell(s_k \circ \sigma) = \begin{cases} \ell(\sigma)+1, & \text{if } \sigma^{-1}(k) < \sigma^{-1}(k+1); \\ \ell(\sigma)-1, & \text{if } \sigma^{-1}(k) > \sigma^{-1}(k+1). \end{cases}$$

[Note: $\sigma^{-1}(i)$ is the position in which the entry i appears in the one-line notation of σ.]

Proof. (a) was essentially shown in the Example above — see (1).

(b) Applying part (a) to σ^{-1} instead of σ, we get

(2) $$\ell(\sigma^{-1} \circ s_k) = \begin{cases} \ell(\sigma^{-1})+1, & \text{if } \sigma^{-1}(k) < \sigma^{-1}(k+1); \\ \ell(\sigma^{-1})-1, & \text{if } \sigma^{-1}(k) > \sigma^{-1}(k+1). \end{cases}$$

But Prop. 5.4 yields $\ell(\sigma^{-1}) = \ell(\sigma)$.

Also, $\ell(\sigma^{-1} \circ \underbrace{s_k}_{=s_k^{-1}}) = \ell(\underbrace{\sigma^{-1} \circ s_k^{-1}}_{=(s_k \circ \sigma)^{-1}}) = \ell((s_k \circ \sigma)^{-1}) \overset{\text{Prop. 5.4}}{=\!=\!=} \ell(s_k \circ \sigma)$.

Thus, (2) transforms into the equality we're proving. $\square$

(For details, see [detnotes, Exercise 4.2 (a)].)

<u>Theorem ~~4.10~~ 5.6.</u> Let $n \in \mathbb{N}$. Let $\sigma \in S_n$. Then, $\ell(\sigma)$ is the minimum $p \in \mathbb{N}$ such that σ can be written as a composition of p simple transpositions (i.e., transpositions of the form s_k).

[Keep in mind: The composition of 0 transpositions is id.]

<u>Example:</u> In S_4, we have

$$[4,1,3,2] = \underbrace{s_2 s_3 s_2}_{= s_3 s_2 s_3} s_1 = s_3 s_2 \underbrace{s_3 s_1}_{= s_1 s_3} = s_3 s_2 s_1 s_3$$

$$= s_2 s_1 s_1 s_3 s_2 s_1 = \cdots .$$

<u>Proof of Thm. 5.6.</u> We need to show:

<u>Claim 1:</u> σ can be written as a composition of $\ell(\sigma)$ simples (= simple transpositions).

<u>Claim 2:</u> σ cannot be written as a composition of $< \ell(\sigma)$ simples.

<u>Proof of Claim 1:</u> Induction on $\ell(\sigma)$.

<u>Base:</u> If $\ell(\sigma)=0$, then $\sigma = \mathrm{id}$, so σ is a composition of 0 simples.

<u>Step:</u> Assume (as the IH) that Claim 1 holds for $\ell(\sigma)=k$. Now, let $\sigma \in S_n$ be such that $\ell(\sigma)=k+1$. Then, $\sigma \neq \mathrm{id}$ (since $\ell(\sigma)=k+1>0$). Hence $\exists\ k \in [n-1]$ such that $\sigma(k) > \sigma(k+1)$. Fix such a σ. Then, Prop. 5.5 (a) yields

$$\ell(\sigma \circ s_k) = \underbrace{\ell(\sigma)}_{=k+1} - 1 = k+1-1 = k.$$

Hence, the IH (applied to $\sigma \circ s_k$ instead of σ) yields that $\sigma \circ s_k$ can be written as a product of k simples:

$$\sigma \circ s_k = s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_k}.$$

Thus,

$$\sigma = s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_h} \circ \underbrace{s_k^{-1}}_{= s_k}$$

$$= s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_h} \circ s_k .$$

This shows that σ is a composition of $h+1 = \ell(\sigma)$ simples,

So ~~the claim~~ Claim 1 holds for $\ell(\sigma) = h+1$.

This completes the inductive proof of Claim 1.

[Underlying idea of the above proof: bubblesort.]

<u>Proof of Claim 2:</u> Prop. 5.5 (a) yields

(3) $\quad \ell(\sigma \circ s_k) \leqslant \ell(\sigma) + 1 \qquad \forall\, \sigma \in S_n$ and $k \in [n-1]$,

Thus, $\forall\, k_1, k_2, \ldots, k_p \in [n-1]$, then

$$\ell(s_{k_1} \circ s_{k_2} \circ \cdots \circ s_{k_p}) \overset{(3)}{\leqslant} \ell(s_{k_1} \circ s_{k_2} \circ \cdots \circ s_{k_{p-1}}) + 1$$

$$\overset{(3)}{\leqslant} \ell(s_{k_1} \circ s_{k_2} \circ \cdots \circ s_{k_{p-2}}) + 2$$

$$\leqslant \cdots \leqslant \underbrace{\ell(\mathrm{id})}_{=0} + p = p.$$

(4)

Now, if σ was the composition of $p < \ell(\sigma)$ simples $s_{k_1}, s_{k_2}, \ldots, s_{k_p}$, then (4) would become $\ell(\sigma) \leq p$, which would contradict $p < \ell(\sigma)$. So Claim 2 is proven. ∎

(For details: [detnotes, Exercise 4.2(g).].) □

Cor. 5.7. Let $n \in \mathbb{N}$.

(a) We have $\ell(\sigma\tau) \equiv \ell(\sigma) + \ell(\tau) \mod 2$ for all $\sigma \in S_n$ and $\tau \in S_n$.

(b) We have $\ell(\sigma\tau) \leq \ell(\sigma) + \ell(\tau)$ for all $\sigma \in S_n$ and $\tau \in S_n$.

(c) If $\sigma = s_{k_1} \circ s_{k_2} \circ \cdots \circ s_{k_q}$, then $q \equiv \ell(\sigma) \mod 2$. □

Proof. [detnotes, Exercises 4.2 and 4.3].

Prop. 5.8. Let $n \in \mathbb{N}$.

(a) We have $\ell(s_k) = 1$, for any $k \in [n-1]$.

(b) We have $\ell(t_{i,j}) = 2|i-j| - 1$ for any $i \neq j$ in $[n]$.

(c) We have $\ell(\mathrm{cyc}_{i,i+1,\ldots,i+k-1}) = k-1$ $\forall i, k$.

(d) We have $\ell(\mathrm{cyc}_{i_1,i_2,\ldots,i_k}) \geq k-1$ $\forall i_1, i_2, \ldots, i_k$ distinct.

Proof. (a) follows from (b),
(b) is [detnotes, Exercise 4.10].
(c), (d) are [detnotes, Exercise 4.16]. $\square$

Prop. 5.9. Let $n \in \mathbb{N}$. Then,

$$\sum_{\omega \in S_n} x^{\ell(\omega)} = (1+x)(1+x+x^2)(1+x+x^2+x^3)\cdots(1+x+x^2+\cdots+x^{n-1})$$

$$= \prod_{i=1}^{n-1} (1+x+\cdots+x^i).$$

Proof. [detnotes, §4.8]. $\square$

5.3. Signs

Def. The *sign* of a permutation $\sigma \in S_n$ (where $n \in \mathbb{N}$) is $(-1)^{\ell(\sigma)}$.
It is called $(-1)^\sigma$ or $\operatorname{sign}(\sigma)$ or $\operatorname{sgn}(\sigma)$ or $\varepsilon(\sigma)$ ....

Thm. 5.10. Let $n \in \mathbb{N}$. Then:

(a) $(-1)^{id} = 1$,

(b) $(-1)^{t_{i,j}} = -1 \quad \forall i \neq j$.

(c) $(-1)^{cyc_{i_1,i_2,\dots,i_k}} = (-1)^{k-1} \quad \forall i_1, i_2, \dots, i_k$ distinct,

(d) $(-1)^{\sigma\tau} = (-1)^{\sigma}(-1)^{\tau} \quad \forall \sigma, \tau \in S_n$.

(e) $(-1)^{\sigma^{-1}} = (-1)^{\sigma} \quad \forall \sigma \in S_n$.

(f) $(-1)^{\sigma\tau\sigma^{-1}} = (-1)^{\tau} \quad \forall \sigma, \tau \in S_n$.

(g) $(-1)^{\sigma} = \prod_{1 \le i < j \le n} \frac{\sigma(i) - \sigma(j)}{i - j} \quad \forall \sigma \in S_n$.

(h) If $x_1, x_2, \dots, x_n$ are any n numbers, and $\sigma \in S_n$, then
$$\prod_{1 \le i < j \le n} (x_{\sigma(i)} - x_{\sigma(j)}) = (-1)^{\sigma} \cdot \prod_{1 \le i < j \le n} (x_i - x_j).$$

Proof. (a) $(-1)^{id} = (-1)^{\ell(id)} = (-1)^0 = 1$,

(b) Prop. 5.8 (b) yields: $\ell(t_{i,j})$ is odd.

(d) $(-1)^{\sigma\tau} = (-1)^{\ell(\sigma\tau)} = (-1)^{\ell(\sigma)+\ell(\tau)}$

(since Cor. 5.7 (a) yields
$\ell(\sigma\tau) \equiv \ell(\sigma) + \ell(\tau) \mod 2$)

$= (-1)^{\ell(\sigma)} (-1)^{\ell(\tau)} = (-1)^{\sigma} (-1)^{\tau}$,

(e) $(-1)^{\sigma^{-1}} = (-1)^{\ell(\sigma^{-1})} = \cancel{k}(-1)^{\ell(\sigma)}$ (by Prop. 5.4)

$= (-1)^{\sigma}$.

(f) $(-1)^{\sigma\tau\sigma^{-1}} = (-1)^{(\sigma\tau)\sigma^{-1}} \overset{(d)}{=} \underbrace{(-1)^{\sigma\tau}}_{\overset{(d)}{=} (-1)^{\sigma}(-1)^{\tau}} \underbrace{(-1)^{\sigma^{-1}}}_{\overset{(e)}{=} (-1)^{\sigma}}$

$= (-1)^{\sigma}(-1)^{\tau}(-1)^{\sigma} = \underbrace{\left((-1)^{\sigma}\right)^2}_{=1} (-1)^{\tau} = (-1)^{\tau}$.

(g), (h) see [detnotes, Exercise 4.13].

(c) Recall from last time:

$\operatorname{cyc}_{i_1, i_2, \ldots, i_k} = t_{i_1, i_2} t_{i_2, i_3} \cdots t_{i_{k-1}, i_k}$.

Hence,

$$(-1)^{cyc_{i_1,i_2,\ldots,i_k}} = (-1)^{t_{i_1,i_2} t_{i_2,i_3} \cdots t_{i_{k-1},i_k}}$$
$$= (-1)^{t_{i_1,i_2}} (-1)^{t_{i_2,i_3}} \cdots (-1)^{t_{i_{k-1},i_k}}$$

(since (d) & induction yield
$$(-1)^{\sigma_1\sigma_2\cdots\sigma_p} = (-1)^{\sigma_1}(-1)^{\sigma_2}\cdots(-1)^{\sigma_p}$$
$\forall \sigma_1,\sigma_2,\ldots,\sigma_p \in S_n$).

$$\underset{(b)}{=} \underbrace{(-1)(-1)\cdots(-1)}_{k-1 \text{ factors}} = (-1)^{k-1}. \qquad \square$$

<u>Def.</u> Let $n \in \mathbb{N}$. A permutation $\sigma \in S_n$ is called <u>even</u> if $(-1)^\sigma = 1$ (i.e., if $\ell(\sigma)$ is even), and <u>odd</u> if $(-1)^\sigma = -1$ (i.e., if $\ell(\sigma)$ is odd).

<u>Cor. 5.11.</u> Let $n \geq 2$. Then,
(# of even $\sigma \in S_n$) = (# of odd $\sigma \in S_n$) $= n!/2$.

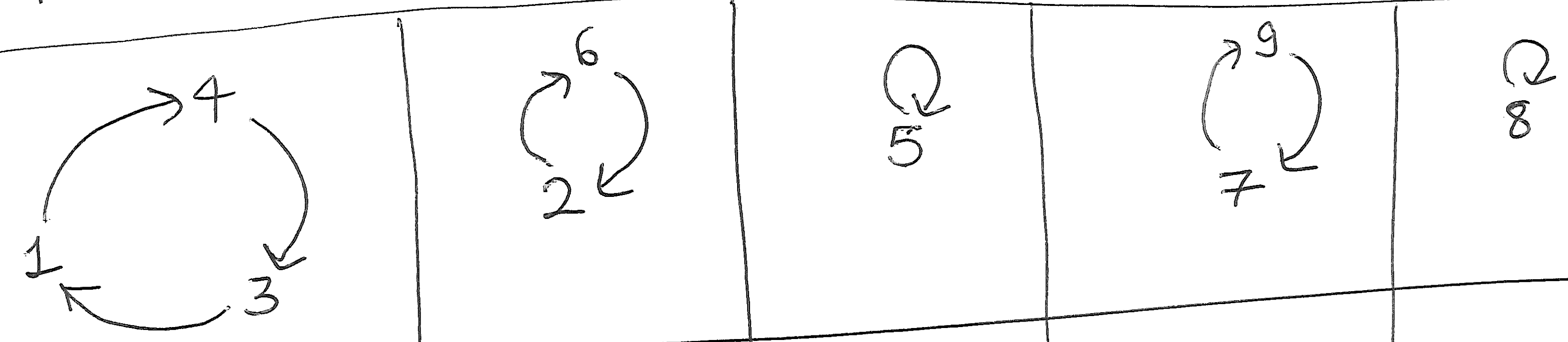
<u>Proof.</u> The map

$$\{\text{even } \sigma \in S_n\} \to \{\text{odd } \sigma \in S_n\},$$
$$\sigma \longmapsto \sigma \circ s_1$$

is a bijection. $\square$

5.4. Cycle decomposition

Example: Let $\sigma = [4,6,1,3,5,2,9,8,7] \in S_9$. Its cycle digraph is



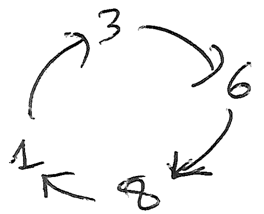
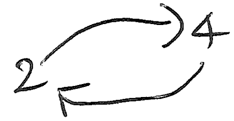
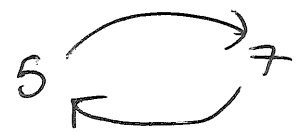
$\operatorname{cyc}_{1,4,3}$	$\operatorname{cyc}_{2,6}$	cyc_{5} $(=\mathrm{id})$	$\operatorname{cyc}_{7,9}$	cyc_{8} $(=\mathrm{id})$

$$\Longrightarrow \quad \sigma = \operatorname{cyc}_{1,4,3} \circ \operatorname{cyc}_{2,6} \circ \operatorname{cyc}_{5} \circ \operatorname{cyc}_{7,9} \circ \operatorname{cyc}_{8}$$

(since the LHS and the RHS act on each $k \in [9]$ in the same way).

Likewise, let $\tau = [3, 4, 6, 2, 7, 8, 5, 1] \in S_8$.

Its cycle digraph is

		
$cyc_{1,3,6,8}$	$cyc_{2,4}$	$cyc_{5,7}$

$$\Longrightarrow \quad \tau = cyc_{1,3,6,8} \circ cyc_{2,4} \circ cyc_{5,7}.$$

Similarly, each $\pi \in S_n$ can be written as a composition of cycles $cyc_{i_1, i_2, \ldots, i_k}$, with each element of $[n]$ appearing in exactly 1 of these cycles.

This representation of π is unique up to swapping the cycles and "cycling each cycle" ($cyc_{i_1, i_2, \ldots, i_k} = cyc_{i_2, i_3, \ldots, i_k, i_1}$

$= cyc_{i_3, i_4, \ldots, i_k, i_1, i_2} = \ldots$).

Thm. 5.12. Let σ be a permutation of a ~~per~~ finite set X,

(a) There is a list $((a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}),$

$(a_{2,1}, a_{2,2}, \ldots, a_{2,n_2}),$

$\ldots\ldots,$

$(a_{k,1}, a_{k,2}, \ldots, a_{k,n_k}))$

of lists of elements of X, such that

- each element of X appears exactly once in $a_{1,1}, a_{1,2}, \ldots, a_{k,n_k}$;

- $\sigma = \operatorname{cyc}_{a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}} \circ \operatorname{cyc}_{a_{2,1}, a_{2,2}, \ldots, a_{2,n_2}} \circ \cdots \circ \operatorname{cyc}_{a_{k,1}, a_{k,2}, \ldots, a_{k,n_k}}$;

Such a list is called a <u>disjoint cycle decomposition</u> of σ.

(b) Any two such lists can be obtained from each other by swapping sublists & cycling each sublist.

(c) If we additionally require

- $a_{1,1} > a_{2,1} > \dots > a_{k,1}$;
- $a_{i,1} \leq a_{i,p}$ $\forall i \; \forall p$,

then this list is unique.

Proof. [Goodman, "Algebra: Abstract & Concrete", proof of Theorem 1.5.3]. Also, see Example above. $\square$