5. PERMUTATIONS

We will go deeper into this subject now, but not too deep.
See [detnotes, Ch. 4] for details, and
[Bóna: Combinatorics of Permutations] for much more.

Recall: a permutation of a set X is a bijection from X to X.

5.1. DEFINITIONS

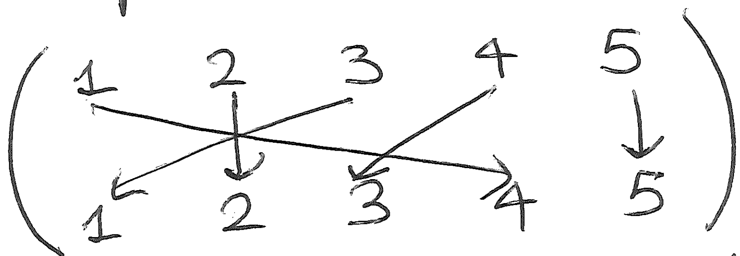
Def. Given $n \in \mathbb{N}$. Let S_n denote the set of all permutations of $[n]$. This set S_n is called the n-th symmetric group. It is closed under composition (i.e., if $\alpha \in S_n$ and $\beta \in S_n$, then $\alpha \circ \beta \in S_n$) and under inverses (i.e., if $\alpha \in S_n$, then $\alpha^{-1} \in S_n$).

Def. Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Then, introduce 2 notations for σ:

(a) The one-line notation of σ is the n-tuple $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$. (The use of square brackets here is a standard.)

Often, one omits the commas & the brackets.

Eg., the permutation



$\in S_5$

~~is written~~ has the one-line notation $[4, 2, 1, 3, 5]$,

or, short, 42135.

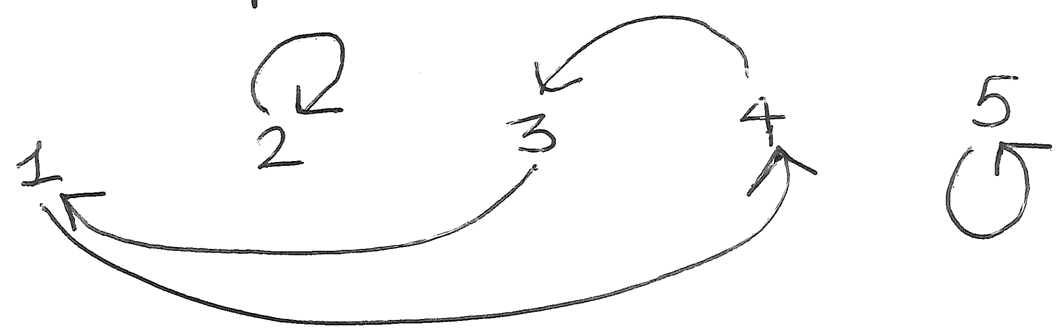
(b) The <u>cycle digraph</u> of σ is defined (informally) as follows:

For each $i \in [n]$, draw a point ("node") labelled i.

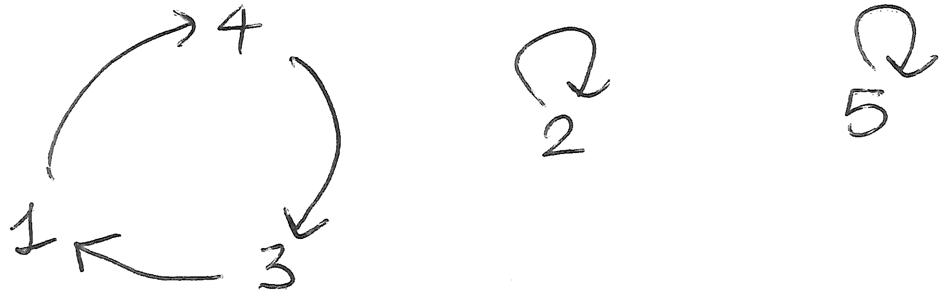
For each $i \in [n]$, draw an ~~curve~~ arrow ("arc") from the node labelled i to the node labelled $\sigma(i)$.

The result is called the cycle digraph of σ.

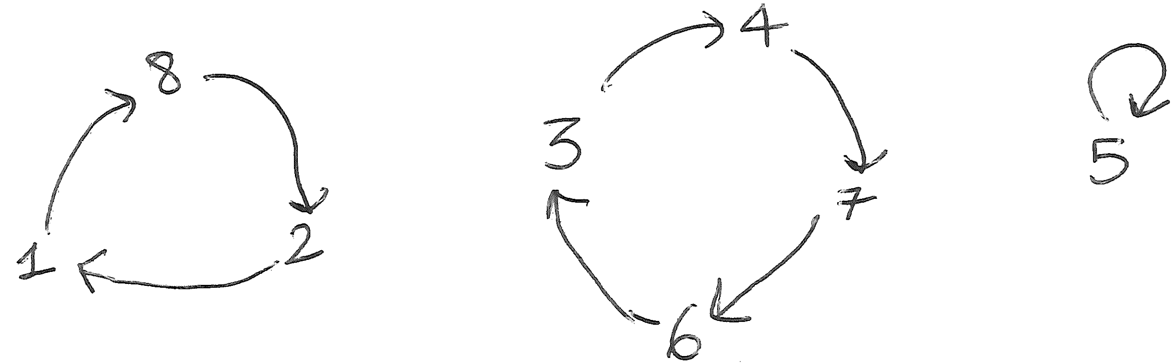
E.g., the above permutation has cycle digraph



or



E.g., the permutation in S_8 whose one-line notation is $[8, 1, 4, 7, 5, 3, 6, 2]$ has cycle digraph



<u>Prop. 5.1.</u> Let $n \in \mathbb{N}$. Then,

$$\begin{aligned} S_n &\longrightarrow \{n\text{-tuples of distinct elements of } [n]\}, \\ \sigma &\longmapsto (\text{one-line notation of } \sigma) = [\sigma(1), \sigma(2), \ldots, \sigma(n)] \end{aligned}$$

is a bijection.

<u>Proof.</u> Permutations of $[n]$ are the same as injective maps $[n] \to [n]$, by the Pigeonhole Principle for Injections. $\square$

5.2. INVERSIONS & LENGTHS

<u>Def.</u> (a) Let i and j be two distinct elements of a set X. Then, the <u>transposition</u> $t_{i,j}$ is the permutation of X that sends i to j, j to i, and leaves everything else in its place.

If $X = [n]$ for some $n \in \mathbb{N}$, then the one-line notation of $t_{i,j}$ is $[1, 2, \ldots, i-1, j, i+1, \ldots, j-1, i, j+1, \ldots, n]$ if $i < j$.

The cycle digraph of $t_{i,j}$ is

(b) Let $n \in \mathbb{N}$ and $i \in [n-1]$. Then the <u>simple transposition</u> <u>$s_i \in S_n$</u> is defined by $s_i = t_{i,i+1}$.

(This was used in Midterm 1 Exercise 2.)

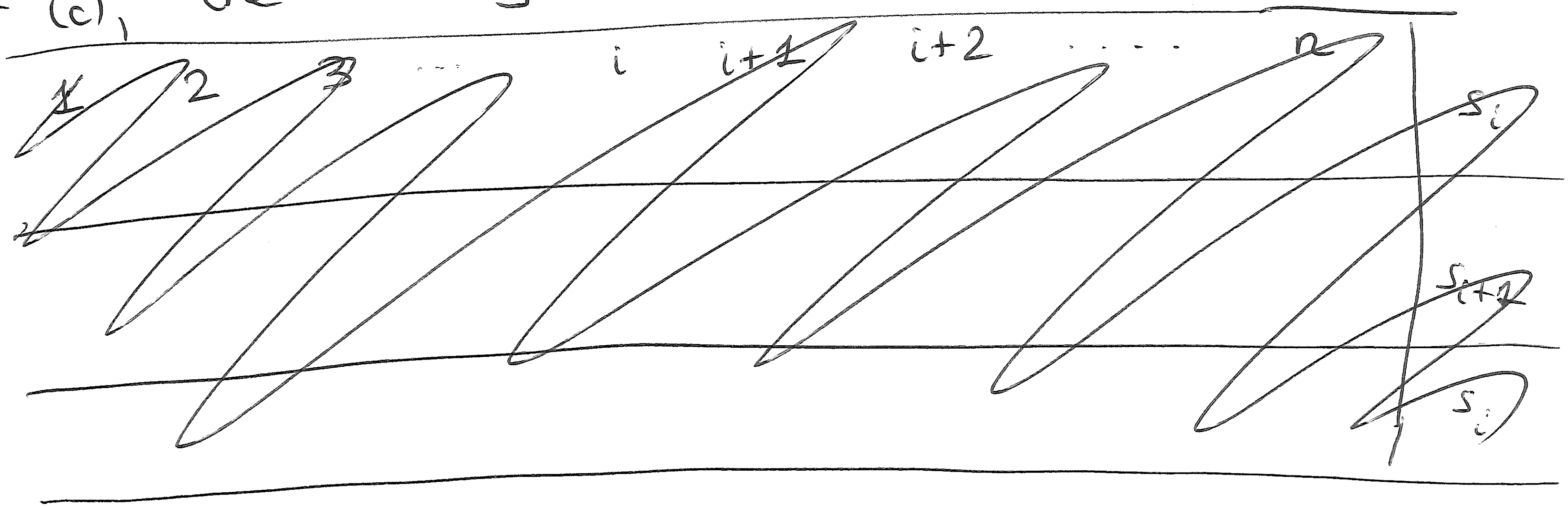
Prop. 5.2. Let $n \in \mathbb{N}$.

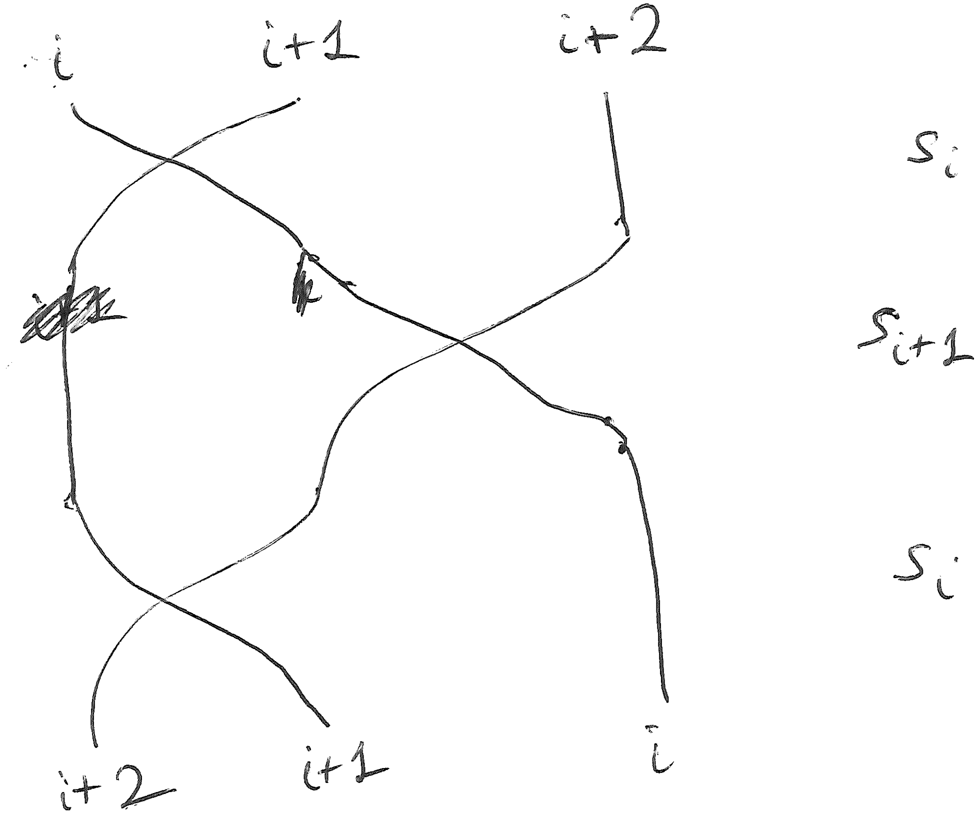
(a) $s_i^2 = id \quad \forall i \in [n-1]$.
(Recall $s_i^2 = s_i \circ s_i$.)

(b) $s_i \circ s_j = s_j \circ s_i \quad \forall i,j \in [n-1]$ with $|i-j|>1$.

(c) $s_i \circ s_{i+1} \circ s_i = s_{i+1} \circ s_i \circ s_{i+1} = t_{i,i+2} \quad \forall i \in [n-2]$.

Proof Everything follows from straightforward casework.
For (c), the following picture helps visualize the argument





□

so $s_i \circ s_{i+1} \circ s_i = t_{i,i+2}$.

Def. Let $n \in \mathbb{N}$. Let w_0 be the permutation in S_n sending each i to $n+1-i$.

In other words, it "reflects" numbers across the middle of $[n]$. It is the unique strictly decreasing permutation of $[n]$.

Example: If $n=5$, then $w_0 = [5,4,3,2,1]$ in one-line notation, with cycle digraph

$$\begin{tikzcd} 1 \arrow[r, bend left] & 5 \arrow[l, bend left] \end{tikzcd} \quad \begin{tikzcd} 2 \arrow[r, bend left] & 4 \arrow[l, bend left] \end{tikzcd} \quad \begin{tikzcd} 3 \arrow[loop above] \end{tikzcd}.$$

If $n=6$, then $w_0 = [6,5,4,3,2,1]$ in one-line notation, with cycle digraph

$$\begin{tikzcd} 1 \arrow[r, bend left] & 6 \arrow[l, bend left] \end{tikzcd} \quad \begin{tikzcd} 2 \arrow[r, bend left] & 5 \arrow[l, bend left] \end{tikzcd} \quad \begin{tikzcd} 3 \arrow[r, bend left] & 4 \arrow[l, bend left] \end{tikzcd}.$$

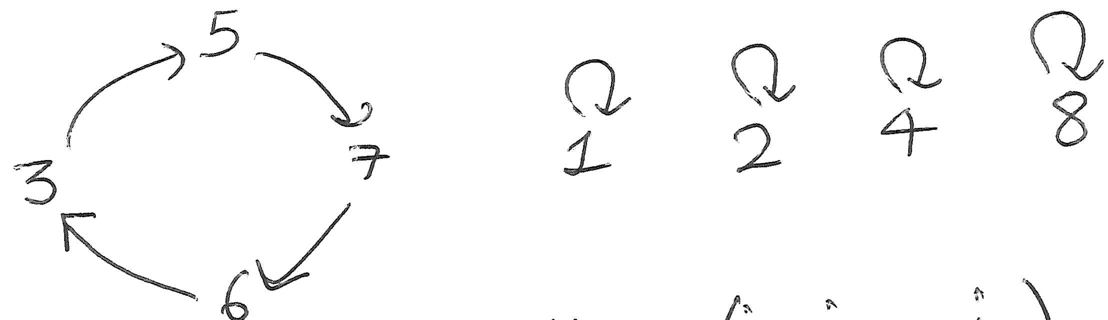
Remark: w_0 and $t_{i,j}$ (for all i,j) are involutions (i.e., permutations σ satisfying $\sigma^2 = \mathrm{id}$).

Def. Let $n \in \mathbb{N}$. Let $i_1, i_2, \ldots, i_k$ be k distinct elements of $[n]$. Then, $\mathrm{cyc}_{i_1, i_2, \ldots, i_k}$ means the permutation in S_n that sends

$i_1 \mapsto i_2,\ i_2 \mapsto i_3,\ \ldots,\ i_{k-1} \mapsto i_k,\ i_k \mapsto i_1,$

and leaves all the other numbers in place.

<u>Example:</u> The cycle digraph of $\mathrm{cyc}_{3,5,7,6} \in S_8$ is



.

<u>Remark:</u> People normally write $(i_1, i_2, \ldots, i_k)$ for $\mathrm{cyc}_{i_1, i_2, \ldots, i_k}$.

<u>Prop. 5.3.</u> Let $n \in \mathbb{N}$.

(a) For any k distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$, we have

$$\mathrm{cyc}_{i_1, i_2, \ldots, i_k} = t_{i_1, i_2} \circ t_{i_2, i_3} \circ \cdots \circ t_{i_{k-1}, i_k}.$$

(b) For any $i \in [n]$ and $k \in \mathbb{N}$ such that $i+k-1 \le n$, we have

$$\mathrm{cyc}_{i, i+1, \ldots, i+k-1} = s_i \circ s_{i+1} \circ \cdots \circ s_{i+k-2}.$$

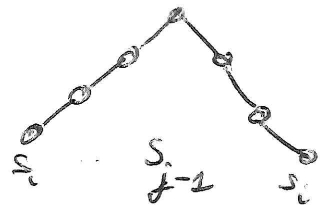
(c) $w_0 = s_1 (s_2 s_1)(s_3 s_2 s_1) \cdots (s_{n-1} s_{n-2} \cdots s_1)$

$= (s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2) s_1$

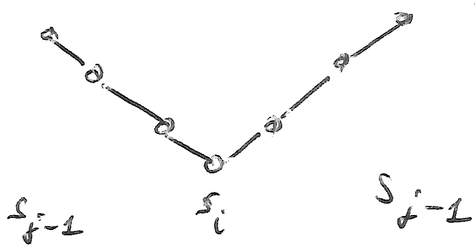
[Here & in the following, $\alpha\beta$ means $\alpha \circ \beta$ when $\alpha, \beta \in S_n$.]

(d) If $1 \le i < j \le n$, then

$t_{i,j} = s_i s_{i+1} \cdots s_{j-1} \cdots s_{i+1} s_i$



$= s_{j-1} s_{j-2} \cdots s_i \cdots s_{j-2} s_{j-1}$



(Both products have $2(j-i)-1$ factors.)

(e) If $i_1, i_2, \ldots, i_k$ are k distinct elements of $[n]$, and if $\sigma \in S_n$, then $\sigma \circ \mathrm{cyc}_{i_1, i_2, \ldots, i_k} \circ \sigma^{-1} = \mathrm{cyc}_{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)}$.

(f) $t_{i,j} = \mathrm{cyc}_{i,j}$.

(g) $s_i = \mathrm{cyc}_{i,i+1}$.

(h) $\mathrm{cyc}_i = \mathrm{id}$.

Proof. (2) Applying $t_{i_1,i_2} \circ t_{i_2,i_3} \circ \cdots \circ t_{i_{k-1},i_k}$ to i_j for some $j \in [k-1]$, we get

$$i_j \xmapsto{t_{i_{k-1},i_k}} i_j \xmapsto{t_{i_{k-2},i_{k-1}}} i_j \cdots\cdots i_j \xmapsto{t_{i_j,i_{j+1}}} i_{j+1} \xmapsto{t_{i_{j-1},i_j}} i_{j+1}$$

$$\cdots\cdots i_{j+1} \xmapsto{t_{i_2,i_3}} i_{j+1} \xmapsto{t_{i_1,i_2}} i_{j+1}.$$

So

$$(t_{i_1,i_2} \circ t_{i_2,i_3} \circ \cdots \circ t_{i_{k-1},i_k})(i_j) = i_{j+1} \quad \forall j \in [k-1].$$
$$= cyc_{i_1,i_2,\ldots,i_k}(i_j)$$

(1)

Applying $t_{i_1,i_2} \circ t_{i_2,i_3} \circ \cdots \circ t_{i_{k-1},i_k}$ to i_k, we get

$$i_k \xmapsto{t_{i_{k-1},i_k}} i_{k-1} \xmapsto{t_{i_{k-2},i_{k-1}}} i_{k-2} \longmapsto \cdots\cdots \xmapsto{t_{i_2,i_1}} i_1.$$

So

$$(t_{i_1,i_2} \circ t_{i_2,i_3} \circ \cdots \circ t_{i_{k-1},i_k})(i_k) = i_1$$

(2) $$= cyc_{i_1,i_2,\ldots,i_k}(i_k).$$

Finally, if $x \notin \{i_1, i_2, \ldots, i_k\}$, then

(3) $$(t_{i_1,i_2} \circ t_{i_2,i_3} \circ \cdots \circ t_{i_{k-1},i_k})(x) = x = cyc_{i_1,i_2,\ldots,i_k}(x).$$

Combining (1), (2) and (3), we see that

$$(t_{i_1,i_2} \circ t_{i_2,i_3} \circ \cdots \circ t_{i_{k-1},i_k})(x) = cyc_{i_1,i_2,\ldots,i_k}(x) \quad \forall x \in [n].$$

Hence, $t_{i_1,i_2} \circ t_{i_2,i_3} \circ \cdots \circ t_{i_{k-1},i_k} = cyc_{i_1,i_2,\ldots,i_k}$.

(b) Apply (a) to $i_1 = i$, $i_2 = i+1$, ..., $i_k = i+k-1$.

(c) For each $k \in [n-1]$, we have $s_1 s_2 \cdots s_k = cyc_{1,2,\ldots,k+1}$

(by part (b), applied to $i=1$).

Now, claim: For each $k \in [n]$, the permutation

$$(s_1 s_2 \cdots s_{k-1})(s_1 s_2 \cdots s_{k-2}) \cdots (s_1 s_2) s_1$$

sends $1, 2, \ldots, k$ to $k, k-1, \ldots, 1$ but leaves $k+1, k+2, \ldots, n$ in their places.

Induction over k:

Induction base ($k=1$): (empty product of permutation) $=\mathrm{id}$.

Induction step ($k \to k+1$):

Assume the permutation

$$\varsigma := (s_1 s_2 \cdots s_{k-1})(s_1 s_2 \cdots s_{k-2}) \cdots (s_1 s_2) s_1$$

sends $1, 2, \ldots, k$ to $k, k-1, \ldots, 1$ while leaving $k+1, k+2, \ldots, n$ in their places. (Induction hypothesis.)

We must prove that the permutation

$$\varsigma' := (s_1 s_2 \cdots s_k)(s_1 s_2 \cdots s_{k-1}) \cdots (s_1 s_2) s_1$$

(4) $$= \underbrace{(s_1 s_2 \cdots s_k)}_{= \mathrm{cyc}_{1,2,\ldots,k+1}} \varsigma = \mathrm{cyc}_{1,2,\ldots,k+1} \varsigma$$

sends $1, 2, \ldots, k+1$ to $k+1, k, \ldots, 1$ while leaving $k+2, \ldots, n$ in their places.

In other words, we must prove

(5) $$\varsigma'(i) = k+2-i \quad \forall i \in [k+1],$$ and

(6) $$\varsigma'(i) = i \quad \forall i > k+1.$$

[Proof of (5): Let $i \in [k+1]$.

If $i \in [k]$, then the induct. hyp. yields $\sigma(i) = k+1-i$, but (4) yields

~~(4) yields~~ $$\sigma'(i) \overset{(4)}{=} \mathrm{cyc}_{1,2,\ldots,k+1}(\underbrace{\sigma(i)}_{=k+1-i}) = \mathrm{cyc}_{1,2,\ldots,k+1}(\underbrace{k+1-i}_{<k+1})$$

$$= (k+1-i)+1 = k+2-i.$$

If $i = k+1$, then

$$\sigma'(i) \overset{(4)}{=} \mathrm{cyc}_{1,2,\ldots,k+1}(\underbrace{\sigma(i)}_{\substack{=\sigma(k+1)=k+1 \\ \text{(by the Ind. hyp.)}}}) = \mathrm{cyc}_{1,2,\ldots,k+1}(k+1)$$

$$= 1 = k+2-\underbrace{(k+1)}_{=i} = k+2-i.$$

So (5) is proven in both cases $i \in [k]$ and $i = k+1$.]

[Proof of (6): LTTR.]

This completes the induction step. $\Rightarrow$ The Claim is proven.

Applying the Claim to $k=n$, we conclude that the permutation $(s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2) s_1$ sends $1, 2, \ldots, n$

to $n, n-1, \ldots, 1$. Thus, it equals w_0. So

(7) $$w_0 = (s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2) s_1 .$$

Next, we want to show

(8) $$w_0 = s_1 (s_2 s_1) \cdots (s_{n-2} s_{n-3} \cdots s_1)(s_{n-1} s_{n-2} \cdots s_1).$$

This is proven similarly, or can be derived from (7) as follows:

Recall $(\alpha_1 \alpha_2 \cdots \alpha_k)^{-1} = \alpha_k^{-1} \alpha_{k-1}^{-1} \cdots \alpha_1^{-1}$ for any k bijective maps $\alpha_1, \ldots, \alpha_k$. So, inverting (7), we get

$$\begin{aligned} w_0^{-1} &= ((s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2) s_1)^{-1} \\ &= s_1^{-1} (s_2^{-1} s_1^{-1}) \cdots (s_{n-2}^{-1} \cdots s_2^{-1} s_1^{-1})(s_{n-1}^{-1} \cdots s_2^{-1} s_1^{-1}) \\ &= s_1 (s_2 s_1) \cdots (s_{n-2} \cdots s_2 s_1)(s_{n-1} \cdots s_2 s_1) \\ &\qquad (\text{since } s_i^{-1} = s_i \ \forall i) \\ &= s_1 (s_2 s_1) \cdots (s_{n-2} s_{n-3} \cdots s_1)(s_{n-1} s_{n-2} \cdots s_1) \end{aligned}$$

$w_0^{-1} = w_0$ (since w_0 is an involution).

So (8) is proven. This proves (c).

The rest is LTTR.

Def. Let $n \in \mathbb{N}$ and $\sigma \in S_n$.

(a) An <u>inversion</u> of σ is a pair (i,j) of elements of $[n]$ such that $i<j$ and $\sigma(i) > \sigma(j)$.

(b) The <u>length</u> of σ is the number of inversions of σ.

<u>Example:</u> Let $\pi = [3,1,4,2] \in S_4$.
The inversions of π are $(1,2)$ (since $1<2$ and $\underset{\underset{3}{\|}}{\pi(1)} > \underset{\underset{1}{\|}}{\pi(2)}$),

$(1,4)$ (since $1<4$ and $\pi(1) > \pi(4)$),

and $(3,4)$.

So the length of π is 3.

<u>Def.</u> The length of a perm. $\sigma \in S_n$ is called $\underline{\ell(\sigma)}$.

<u>Remark:</u> If $\sigma \in S_n$, then $0 \leq \ell(\sigma) \leq \binom{n}{2}$.

The only $\sigma \in S_n$ with $\ell(\sigma) = 0$ is id.

(Indeed, if $\ell(\sigma) = 0$, then ~~every~~ σ has no inversions, so $\forall i<j$ satisfy $\sigma(i) \leq \sigma(j)$,

so $\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(n)$, so

$\sigma(1) < \sigma(2) < \cdots < \sigma(n)$, so $\sigma = \mathrm{id}$.)

The only $\sigma \in S_n$ with $\ell(\sigma) = \binom{n}{2}$ is w_0.

Inbetween, there are many:

S_3:

(permuts. in one-line notation)

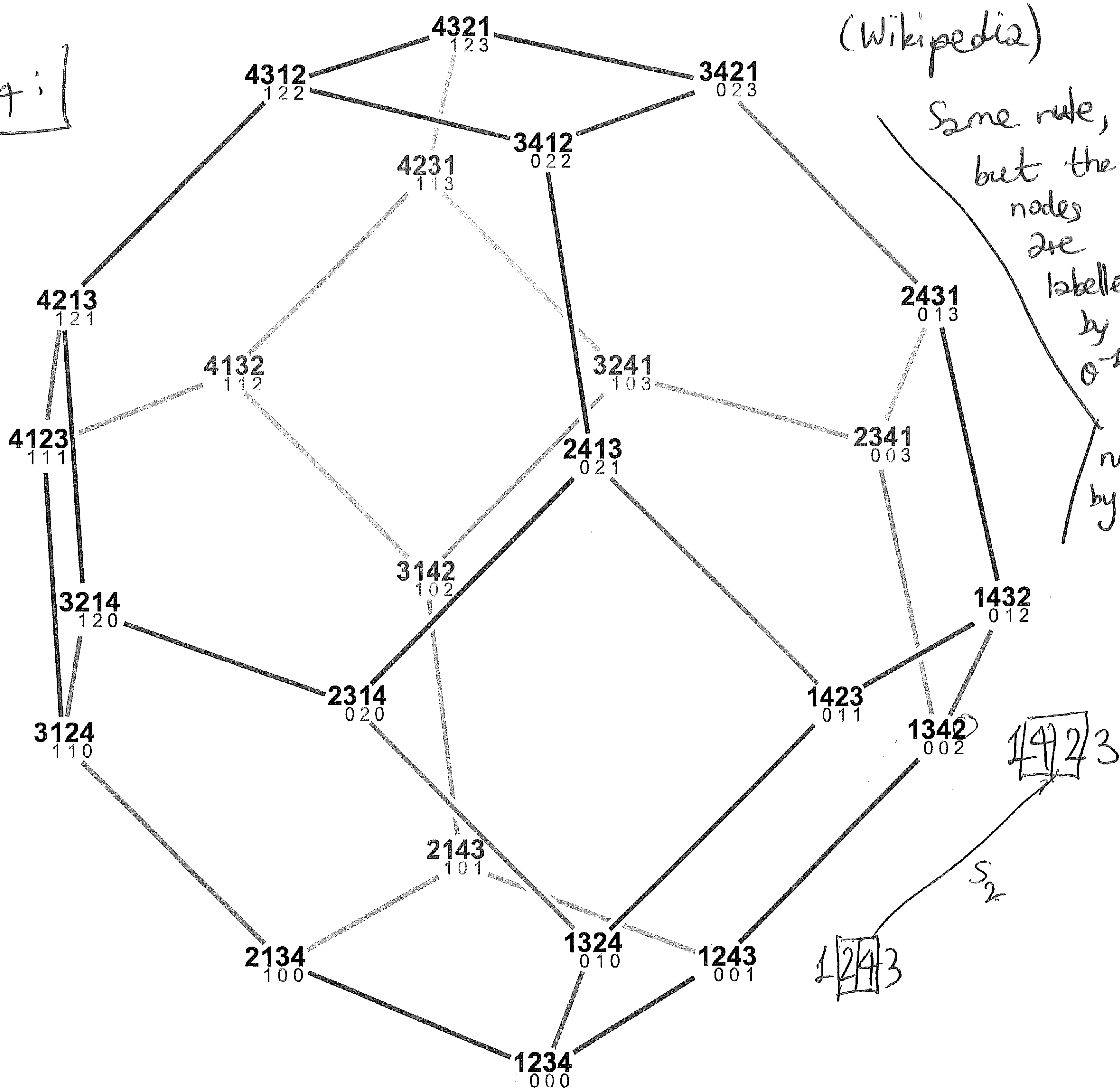
$$\begin{tikzcd}
 & 321 = w_0 \arrow[dl, dash, "s_1"'] \arrow[dr, dash, "s_2"] & & \leftarrow \ell = 3 \\
231 \arrow[d, dash, "s_2"] & & 312 \arrow[d, "s_1"] & \leftarrow \ell = 2 \\
213 \arrow[dr, dash, "s_1"'] & & 132 \arrow[dl, dash, "s_2"] & \leftarrow \ell = 1 \\
 & 123 = \mathrm{id} & & \leftarrow \ell = 0
\end{tikzcd}$$

We draw an edge
$$\alpha \xrightarrow{s_i} \beta$$
if $\alpha = \beta \circ s_i$ (or, equivalently, $\beta = \alpha \circ s_i$).

S_4:

(Wikipedia)

Same rule, but the nodes are labelled by σ^{-1}, not by σ.



The "permutohedron"

To each $\sigma \in S_n$, assign the point $(\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(n)) \in \mathbb{R}^n$.

Then, these $n!$ points are the vertices of the permutohedron.