The twelvefold way so far:

$A \to X$ / f is...	arbitrary	injective	surjective
$L \to L$	$\lvert X\rvert^{\lvert A\rvert}$	$\lvert X\rvert(\lvert X\rvert-1)(\lvert X\rvert-2)\cdots(\lvert X\rvert-\lvert A\rvert+1)$	$sur(\lvert A\rvert, \lvert X\rvert) = \lvert X\rvert! \left\{ {\lvert A\rvert \atop \lvert X\rvert} \right\}$
$U \to L$	$\binom{\lvert A\rvert+\lvert X\rvert-1}{\lvert A\rvert}$	$\binom{\lvert X\rvert}{\lvert A\rvert}$	$\binom{\lvert A\rvert-1}{\lvert A\rvert-\lvert X\rvert}$
$L \to U$	$\left\{ {\lvert A\rvert \atop 0} \right\} + \left\{ {\lvert A\rvert \atop 1} \right\} + \dots + \left\{ {\lvert A\rvert \atop \lvert X\rvert} \right\}$	$[\lvert A\rvert \leq \lvert X\rvert]$	$\left\{ {\lvert A\rvert \atop \lvert X\rvert} \right\}$
$U \to U$			

More on Stirling numbers of 2nd kind:

(Recall: If $n \in \mathbb{N}$ and $k \in \mathbb{N}$, then $\left\{ {n \atop k} \right\} = \frac{1}{k!} \operatorname{sur}(n,k)$ is the # of all set partitions of $[n]$ into k parts.)

Prop. 4.11. Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$.

(a) We have $\left\{ {n \atop 0} \right\} = [n=0]$.

(b) $\left\{ {0 \atop k} \right\} = [k=0]$.

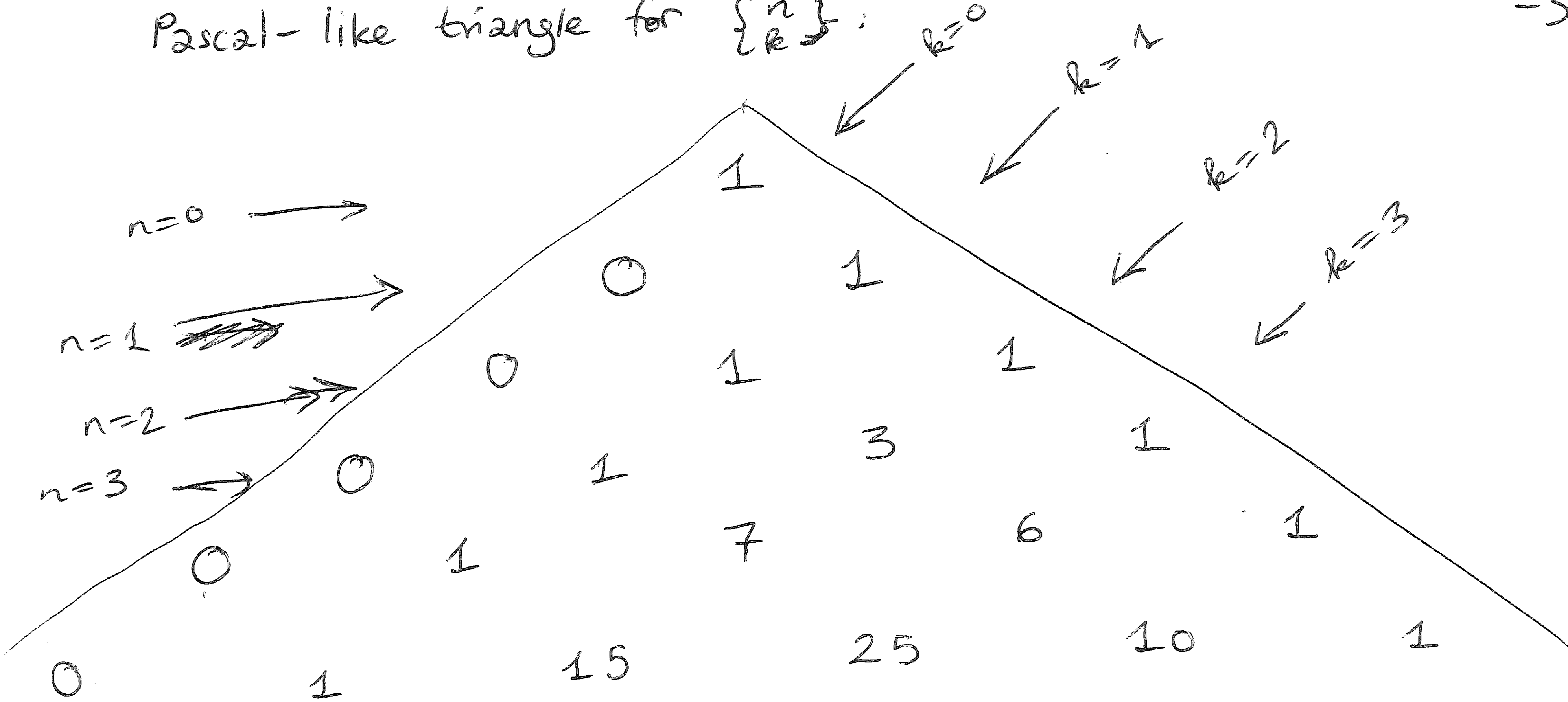
(c) $\left\{ {n \atop k} \right\} = 0$ if $k > n$.

(d) $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$ if $n>0$ & $k>0$.

(e) $\left\{ {n \atop k} \right\} = \sum\limits_{j=0}^{n-1} \binom{n}{j} \left\{ {j \atop k-1} \right\} / k$.

Proof. Follows from Prop. 3.10, Prop. 3.11, Prop. 3.12. $\square$

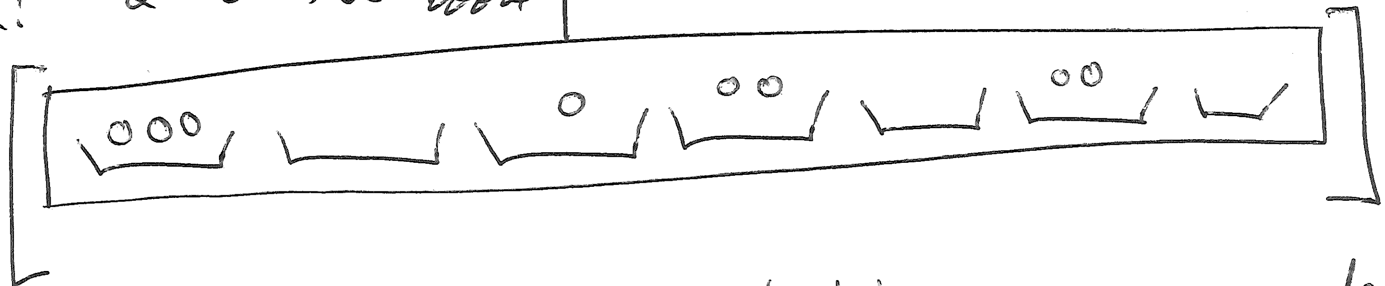
Pascal-like triangle for $\left\{ {n \atop k} \right\}$:



<u>Note:</u> If p is prime, then $p \mid \left\{ {p \atop k} \right\}$ $\forall k \in \{2,3,\dots,p-1\}$. Can you prove it? (Compare to $p \mid \binom{p}{k}$ $\forall k \in \{1,2,\dots,p-1\}$, but this is harder.)

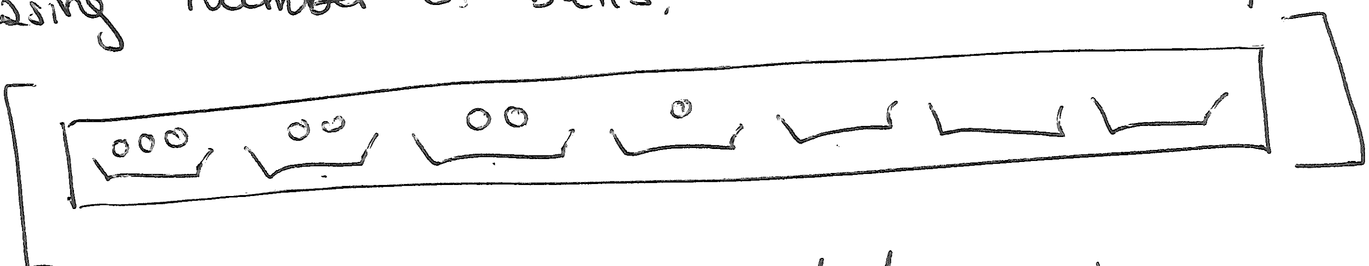
4.6. $U \to U$, AND INTEGER PARTITIONS

Idea: a $U \to U$ ~~[illegible]~~ placement has the form



box $\sim$ ball

Since the boxes are indistinguishable, you can order them by decreasing number of balls. So the above placement is



box $\sim$ ball.

Such a placement is thus encoded by how many balls each box has: here, the numbers are $3, 2, 2, 1, 0, 0, 0$.
The decreasing order makes this encoding unique.

<u>Def.</u> A <u>partition</u> of an integer n is a weakly decreasing list $(a_1, a_2, \ldots, a_k)$ of positive integers whose sum is n (that is, $a_1 \geq a_2 \geq \cdots \geq a_k > 0$ and $a_1 + a_2 + \cdots + a_k = n$).

The integers $a_1, a_2, \dots, a_k$ are called the <u>parts</u> of the partition.

<u>Example:</u> The partitions of 5 are

(5), $(4,1)$, $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$, $(1,1,1,1,1)$.

The only partition of 0 is $()$.

(So a partition is a weakly decreasing composition.)

<u>Def.</u> Let $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then, <u>$p_k(n)$</u> means the # of partitions of n into k parts (=having k parts).

<u>Example:</u> $p_0(5) = 0$, $p_1(5) = 1$, $p_2(5) = 2$, $p_3(5) = 2$, $p_4(5) = 1$, $p_5(5) = 1$, $p_k(5) = 0 \ \forall k > 5$.

Prop. 4.12. (a) $p_k(n) = 0$ $\forall n < 0$,

(b) $p_k(n) = 0$ if $k > n$.

(c) $p_0(n) = [n=0]$.

(d) $p_1(n) = [n \neq 0]$.

(e) $p_k(n) = p_k(n-k) + p_{k-1}(n-1)$ $\forall k \geq 1$ and $n \in \mathbb{Z}$.

(f) $p_2(n) = \lfloor n/2 \rfloor$ (floor function).

Proof. (a) A sum of positive integers is not negative.

(b) A partition into k parts has sum $\geq \underbrace{1+1+\dots+1}_{k \text{ times}} = k$.

(c) The only partition into 0 parts is $()$.

(d) The only partition of n into 1 parts is (n), which exists only if $n \neq 0$.

(e) Classify the partitions of n into k parts into 2 types:

Type 1: partitions whose last entry is 1.

Type 2: partitions whose last entry is $\neq 1$
(i.e., partitions whose all entries are ≥ 2).

Now, there is a bijection

$$\{\text{Type-1 partitions of } n \text{ into } k \text{ parts}\} \to \{\text{partitions of } n-1 \text{ into } k-1 \text{ parts}\},$$

$$(\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, 1) \longmapsto (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}).$$

Thus, $\#(\text{Type-1 partitions}) = p_{k-1}(n-1)$.

Also, there is a bijection

$$\{\text{Type-2 partitions of } n \text{ into } k \text{ parts}\} \to \{\text{partitions of } n-k \text{ into } k \text{ parts}\},$$

$$(\lambda_1, \lambda_2, \ldots, \lambda_k) \longmapsto (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_k - 1).$$

Thus, $\#(\text{Type-2 partitions}) = p_k(n-k)$.

Adding these equalities together, we get the claim of (e).

(f) The partitions of n into 2 parts are

$$(n-1, 1),\ (n-2, 2),\ (n-3, 3), \ldots, \left(\left\lceil \tfrac{n}{2} \right\rceil, \left\lfloor \tfrac{n}{2} \right\rfloor\right).$$

ceiling function

$\square$

By the way...

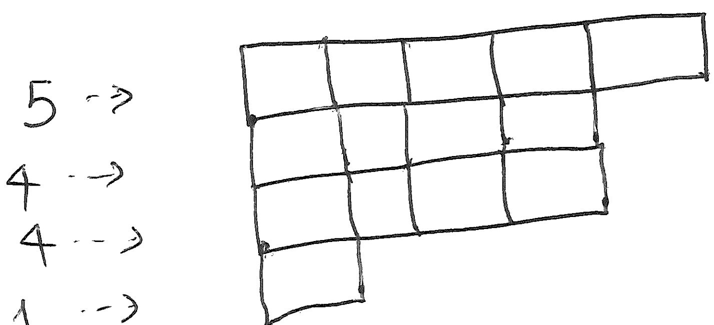
__Prop. 4.13.__ Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then,

$p_k(n)$ = # of partitions of n whose largest part is k.

__Proof outline.__ Picture proof: e.g., let $k=4$ and $n = 14$,

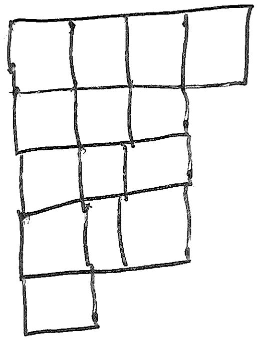
Start with the partition $(5, 4, 4, 1)$ of n into k parts.

~~Draw~~ Draw a table of k left-aligned rows, where the length of each row is the corresponding part of the partition.



(This table is called a __Young diagram__ or __Ferrers diagram__.)

Now, flip this table around the \ diagonal.



The length of the rows of this new table ~~are the~~ form a partition of n whose largest part is k.
(here, $(4,3,3,3,1)$).

This is a bijection.

(It is called <u>conjugation</u>.) $\square$

<u>Prop. 4.14.</u> (# of surjective $U \to U$ placements) $= p_{|X|}(|A|)$.

<u>Proof outline.</u> Encode a surjective $U \to U$ placement as a partition of $|A|$ into $|X|$ parts (where each part is the # of balls in some box). This is a bijection. $\square$

Prop. 4.15. (# of $U \to U$ placements)

$$= p_0(|A|) + p_1(|A|) + \cdots + p_{|X|}(|A|).$$

$\square$

Proof. Exercise.

Prop. 4.16. (# of injective $U \to U$ placements) $= [|A| \leq |X|]$.

$\square$

Proof. Exercise.

The twelvefold way: ~~[illegible]~~

f is... / $A \to X$	arbitrary	injective	surjective
$L \to L$	$\lvert X\rvert^{\lvert A\rvert}$	$\lvert X\rvert(\lvert X\rvert-1)(\lvert X\rvert-2) \cdots (\lvert X\rvert-\lvert A\rvert+1)$	$sur(\lvert A\rvert, \lvert X\rvert)$ ← $= \lvert X\rvert! \left\{ {\lvert A\rvert \atop \lvert X\rvert} \right\}$
$U \to L$	$\binom{\lvert A\rvert+\lvert X\rvert-1}{\lvert A\rvert}$	$\binom{\lvert X\rvert}{\lvert A\rvert}$	$\binom{\lvert A\rvert-1}{\lvert A\rvert-\lvert X\rvert}$
$L \to U$	$\left\{ {\lvert A\rvert \atop 0} \right\} + \left\{ {\lvert A\rvert \atop 1} \right\} + \dots + \left\{ {\lvert A\rvert \atop \lvert X\rvert} \right\}$	$[\lvert A\rvert \leqslant \lvert X\rvert]$	$\left\{ {\lvert A\rvert \atop \lvert X\rvert} \right\}$
$U \to U$	$p_0(\lvert A\rvert) + p_1(\lvert A\rvert) + \dots + p_{\lvert X\rvert}(\lvert A\rvert)$	$[\lvert A\rvert \leqslant \lvert X\rvert]$	$p_{\lvert X\rvert}(\lvert A\rvert)$

4.7. Odds & Ends

Example: Given ~~#~~ n persons $(n>0)$ and k tasks $(k>0)$.

(a) What is the # of ways to assign to each person a task such that each task has at least one person working on it?

(b) What if we additionally want to choose a leader for each task (among the people assigned to it)?

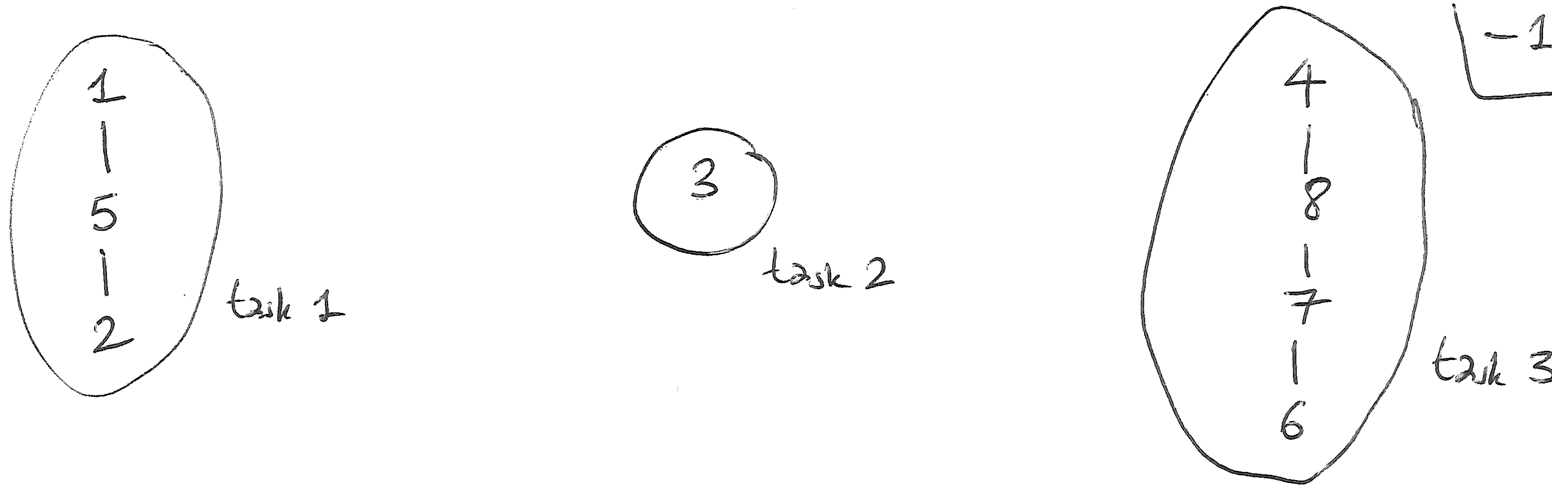
(c) What if, instead, we want to choose a vertical hierarchy (between the people working on the task) for each task?

Example: 8 people $(1, 2, 3, 4, \dots, 8)$, and 3 tasks.

(a) 1 2 5 task 1 | 3 task 2 | 4 6 7 8 task 3

(b) 1 $\boxed{2}$ 5 task 1 | $\boxed{3}$ task 2 | 4 6 $\boxed{7}$ 8 task 3

(c)



Answers:

(a) $\mathrm{sur}(n,k)$.

(b) $(n(n-1)\cdots(n-k+1))\cdot k^{n-k}$.

(c) $n!\cdot\binom{n-1}{k-1}$.

(See 4909 Fall 2017 Oct 10.)

4.8. A GLIMPSE OF RANKING & UNRANKING

Ranking & unranking is an important subject; we only will scratch the surface. See [Loehr, Ch. 5] and [Knuth, "The Art of Computer Programming", vol. 4A] for more.

<u>Statement of the problem:</u> You have a finite set X. Find ways to

(1) construct a list of all elements of X ("listing").

(2) for each $k \in [|X|]$, compute the k-th element of this list <u>without</u> having to write down the whole list ("unranking").

(3) for each $x \in X$, compute the <u>rank</u> of x in this list (i.e., the # of elements before x in the list) ("ranking").

Equivalently: construct a bijection $[|X|] \to X$ and its inverse.
("construct" = find an algorithm.)

Same questions apply to a countable set X, using $\{1, 2, 3, \ldots\}$ instead of $[|X|]$.

<u>Example 1:</u> Let $N \in \mathbb{N}$. Let $\cancel{\mathcal{P}([N])}$ $\mathcal{P}(X)$ (where X is any set) denote the set of all subsets of X.

How to list $\mathcal{P}([N])$ (a 2^N-element set)?

<u>Example:</u> A list for $\mathcal{P}([3])$:

$(\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \varnothing, \{1,2,3\})$.

This is not very systematic.

Better: recursion.

A list of $\mathcal{P}([0])$: $(\varnothing)$.

A list of $\mathcal{P}([N+1])$ from a list of $\mathcal{P}([N])$:

To build a list of $\mathcal{P}([N+1])$ from a list of $\mathcal{P}([N])$, we first write down this list of $\mathcal{P}([N])$, and then write down a second copy of this list, but replacing each subset S by $S \cup \{N+1\}$.

For example:

$\mathcal{P}([0])$	$(\varnothing)$
$\mathcal{P}([1])$	$(\varnothing, \{1\})$
$\mathcal{P}([2])$	$(\varnothing, \{1\}, \{2\}, \{1,2\})$
$\mathcal{P}([3])$	$(\varnothing, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\})$

Now, how do we rank/unrank in this list?

Observations:

- The list of $\mathcal{P}([N+1])$ starts with the list of $\mathcal{P}([N])$. Thus, the rank of a $S \in \mathcal{P}([N])$ does not depend on N.
- If $S \in \mathcal{P}([N])$ has rank r, then $S \cup \{N+1\} \in \mathcal{P}([N+1])$ has rank $r + 2^N$.

$\Longrightarrow$ (by induction): The rank of any $S \in \mathcal{P}([N])$ is

$$\sum_{i \in S} 2^{i-1}.$$

(E.g., the rank of $\{1,3\}$ is $2^{1-1} + 2^{3-1} = 1 + 4 = 5$.)

To unrank, we start with an $r \in \{0, 1, \ldots, 2^N - 1\}$, and we look for the subset S of $[N]$ with $\sum_{i \in S} 2^{i-1} = r$.

This is just the set of positions of the 1-bits in the binary expansion of r.

<u>Exercise:</u> Ranking/unranking for lacunar subsets leads to the Zeckendorf theorem.

Example 2: Given $j \in \mathbb{N}$ and a set X, we let $\mathcal{P}_j(X)$ be the set of all j-element subsets of X.

Given $j \in \mathbb{N}$, how to list $\mathcal{P}_j([N])$ or $\mathcal{P}_j(\mathbb{N})$?

Example: $j=3$; try to list $\mathcal{P}_j(\mathbb{N})$.

Bad attempt: $\{0,1,2\}$, $\{0,1,3\}$, $\{0,1,4\}$, ...

This never gets to $\{0,2,3\}$.

Better: recursion. Let's first list $\mathcal{P}_j(\{0,1,..,N\})$ for each $N \geq -1$.

Again, $\mathcal{P}_0$(anything) has list $(\emptyset)$.

Also, $\mathcal{P}_j(\{0,1,..,-1\})$ has list $()$ if $j>0$.

To build a list of $\mathcal{P}_j(\{0,1,..,N+1\})$ from a list of $\mathcal{P}_j(\{0,1,..,N\})$ and a list of $\mathcal{P}_{j-1}(\{0,1,..,N\})$, we first write down the list of $\mathcal{P}_j(\{0,1,..,N\})$, and then write down the list of $\mathcal{P}_{j-1}(\{0,1,..,N\})$, but replacing each subset S by $S \cup \{N+1\}$.

For example:

$\mathcal{P}_0(\emptyset)$	$(\emptyset)$
$\mathcal{P}_0(\{0\})$	$(\emptyset)$
$\mathcal{P}_1(\{0\})$	$(\{0\})$
$\mathcal{P}_0(\{0,1\})$	$(\emptyset)$
$\mathcal{P}_1(\{0,1\})$	$(\{0\}, \{1\})$
$\mathcal{P}_2(\{0,1\})$	$(\{0,1\})$
$\mathcal{P}_0(\{0,1,2\})$	$(\emptyset)$
$\mathcal{P}_1(\{0,1,2\})$	$(\{0\}, \{1\}, \{2\})$
$\mathcal{P}_2(\{0,1,2\})$	$(\{0,1\}, \{0,2\}, \{1,2\})$
$\mathcal{P}_3(\{0,1,2\})$	$(\{0,1,2\})$
$\vdots$	
$\mathcal{P}_2(\{0,1,2,3\})$	$(\{0,1\}, \{0,2\}, \{1,2\}, \{0,3\}, \{1,3\}, \{2,3\})$

...

Again, the list of $\mathcal{P}_j(\{0,1,..,N+1\})$ starts with the list of $\mathcal{P}_j(\{0,1,..,N\})$. Thus, the rank of an $S \in \mathcal{P}_j(\{0,1,..,N\})$ does not depend on N.

How to rank/unrank?

Observe:

- If $S \in \mathcal{P}_{j-1}(\{0,1,..,N\})$ has rank r, then $S \cup \{N+1\}$ has rank $\binom{N+1}{j} + r$.

$\Rightarrow$ (by induction): The rank of any $\{s_1 < s_2 < \dots < s_j\} \in \mathcal{P}_j(\{0,1,..,N\})$ is $\binom{s_1}{1} + \binom{s_2}{2} + \dots + \binom{s_j}{j}$.

This solves Exercise 7 on Midterm 1 again, as the list of $\mathcal{P}_j(\mathbb{N})$ is infinite in length when $j>0$.

To unrank, you need to expand an $n \in \mathbb{N}$ in the form $n = \binom{s_1}{1} + \binom{s_2}{2} + \dots + \binom{s_j}{j}$ with $0 \le s_1 < s_2 < \dots < s_j$.