4. THE TWELVEFOLD WAY

4.1. WHAT IS IT?

The twelvefold way is a table of $4 \cdot 3 = 12$ standard counting problems that frequently appear.

Informal description: Given a set A of balls, and a set X of boxes.

A placement ~~##~~ means a way to distribute the balls into the boxes.

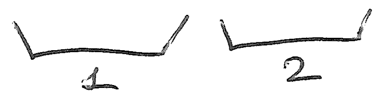
Rigorously: a placement is a map from A to X.

At least, this is what will be called the "L→L placements".

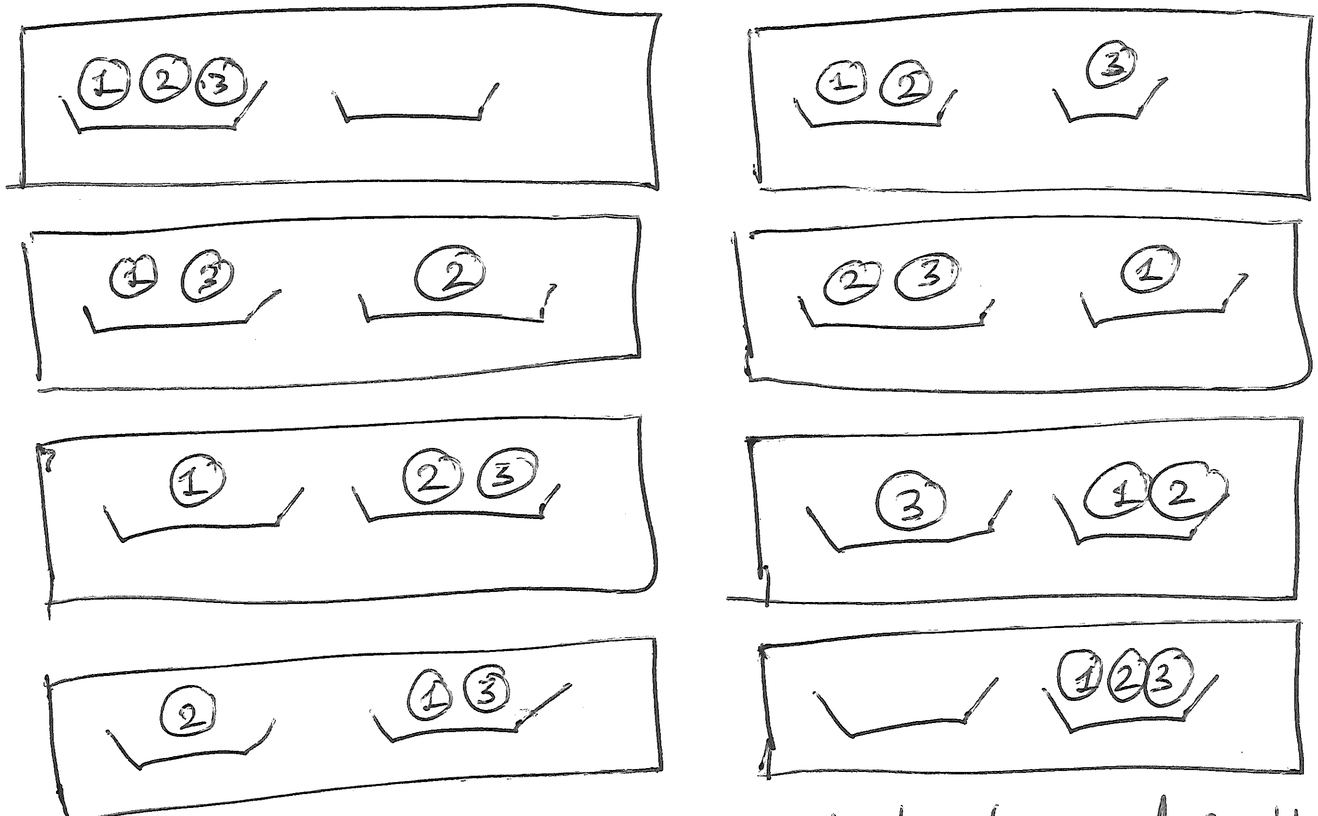
How many placements are there? $|X|^{|A|}$.

Example: $|X| = 2$, $|A| = 3$.

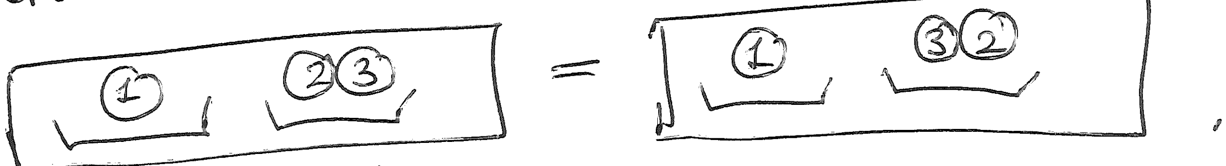
For example, take $X = \{1,2\}$ and $A = \{1,2,3\}$.

Always draw boxes in increasing order: 

Here are the 8 L→L placements:



The order of the balls in a single box doesn't matter:

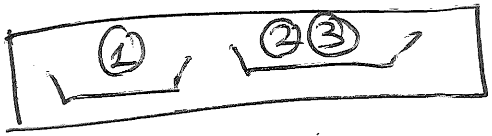


This suggests the following variations:

- What if we require $f : A \to X$ to be injective (i.e., each box contains ≤ 1 ball), or surjective (i.e., each box contains ≥ 1 ball)?

- What if the balls are unlabelled (i.e., indist-inguishable)?

~~What if Note~~
We have not made this rigorous, but the gist is that we treat, e.g.,



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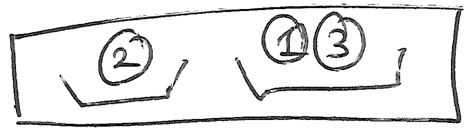
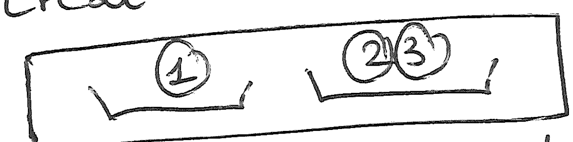


as the same placements.
We will see how to make this rigorous.

- What if the boxes are unlabelled? I.e., what if we treat



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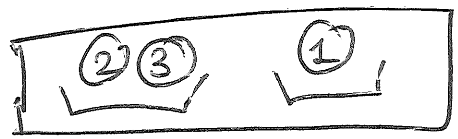


as the same placement?

- What if both balls and boxes are unlabelled?

So we get $3 \cdot 4 = 12$ different counting problems.
List them as a table:

$A \to X$ \ f is...	arbitrary	injective	surjective
$L \to L$	$\lvert X \rvert^{\lvert A \rvert}$		
$U \to L$			
$L \to U$			
$U \to U$			

For example, $L \to U$ means "balls are labelled, boxes are unlabelled".

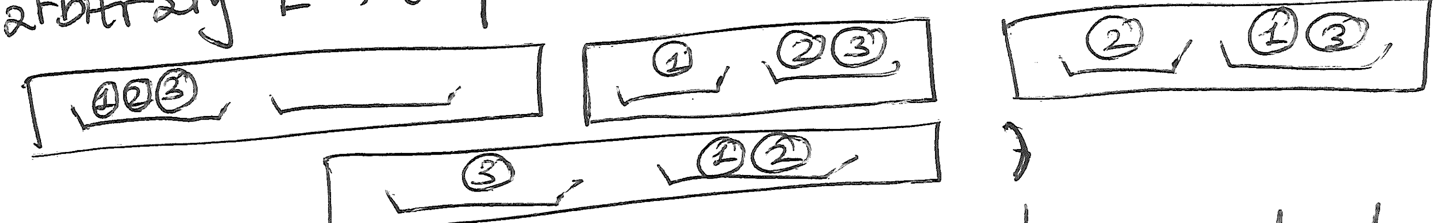
The goals of this chapter are:

- Formalize $U \to L$, $L \to U$ and $U \to U$.
- Fill in the remaining 11 cells.
- See some examples.

Example: $|X| = 2$, $|A| = 3$.

$A \to X$ \ f is...	arbitrary	injective	surjective
L → L	8	0 (since $\lvert X\rvert < \lvert A\rvert$)	6
U → L	4	0	2
L → U	4	0	3
U → U	2	0	1

(e.g., arbitrary L → U placements:



)

In general: Not each of the 12 questions has a closed-form solution. But there are good recursive answers.

4.2. L→L

L→L placements are just maps $A \to X$.

Prop. 4.1. (# of L→L placements $A \to X$) $= |X|^{|A|}$.

Proof. This is Theorem 3.4. $\square$

Prop. 4.2. (# of injective L→L placements $A \to X$)

$= $ (# of injective maps $A \to X$)

$= |X|(|X|-1)(|X|-2)\cdots(|X|-|A|+1)$.

Proof. This is Theorem 3.5. $\square$

Prop. 4.3. (# of surjective L→L placements $A \to X$)

$= $ (# of surjective maps $A \to X$)

$= \mathrm{sur}(|A|, |X|)$.

Proof. This is Proposition 3.9. $\square$

(See HW3 exercise 2 for a formula for $\mathrm{sur}(n,k)$.)

Typical applications of $L \to L$ placements:

- assigning grades (from a finite set X) to students (from a finite set A):

  $L \to L$ placements (arbitrary),

- assigning IP addresses to a bunch of computers:

  injective $L \to L$ placements,

- How many 8-digit telephone numbers are there with no 2 equal digits?

  injective $L \to L$ placements (with $A = [8]$ and $X = \{0, 1, .., 9\}$)

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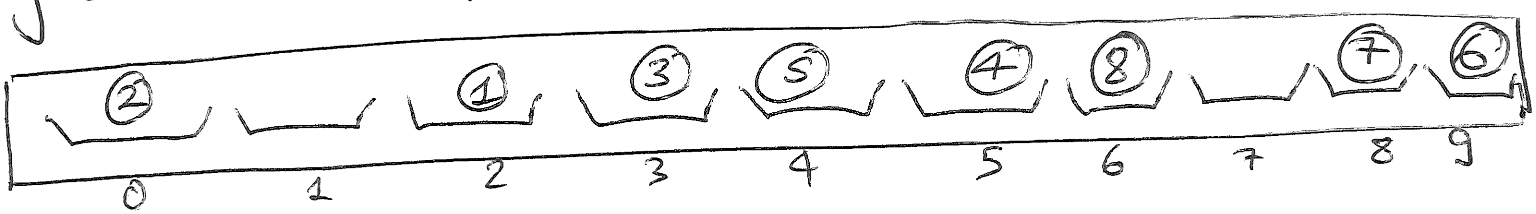


$\triangleq$ telephone number 20354986)

$\Rightarrow$ the number of such numbers is $10 \cdot 9 \cdot \ldots \cdot 3 = \frac{10!}{2!}$.

Remark: Here's a quick problem NOT from the twelvefold way:

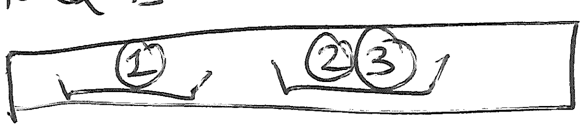
How many 8-digit telephone numbers contain no 2 adjacent equal digits?
(e.g. 31315315 is okay, but 12334567 isn't.)

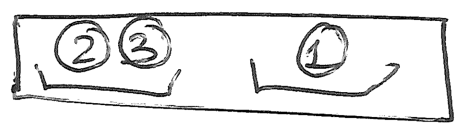
Answer:

$$\underset{\substack{\text{options}\\ \text{for 1st}\\ \text{digit}}}{10} \cdot \underset{\substack{\text{options}\\ \text{for 2nd}\\ \text{digit}}}{9} \cdot \underset{\substack{\text{options}\\ \text{for 3rd}\\ \text{digit}}}{9} \cdot 9 \cdot \ldots \cdot 9$$

$$= 10 \cdot 9^7.$$

4.3. UNLABELLED OBJECTS

What does it mean for balls, or boxes, to be unlabelled?
The idea is that (with unlabelled boxes) we want to treat 

and 

as the

same.

The rigorous way to do this is by introducing an equivalence relation & passing to equivalence classes.

(References for equivalence classes: notes/~~[illegible]~~ slides by Melissa Lynn, as linked from the class website.)

<u>Def.</u> A <u>(binary) relation</u> on a set S is a subset of $S \times S$,

Idea: something like $=$ or $\leq$ or $\subseteq$ or $|$ (divides), or $\vdots$ (divisible by), or $\cong$, or $\neq$, or ~~[illegible]~~ "$\equiv$ mod k" for a given $k \in \mathbb{Z}$, or $\sim$ for geometric shapes, ... (many more).

If R is a relation on S, then we write aRb if and only if $(a,b) \in R$.

For example, the relation $\leq$ on $\mathbb{N}$ is really the set of all $(a,b) \in \mathbb{N} \times \mathbb{N}$ with $a \leq b$.

Def. An <u>equivalence relation</u> on a set S is

a relation $\sim$ which is:

- reflexive. (i.e., it satisfies $a \sim a \ \forall a \in S$);
- symmetric (i.e., if $a \sim b$, then $b \sim a$);
- transitive (i.e., if $a \sim b$ and $b \sim c$, then $a \sim c$).

Idea: An equivalence relation relates objects that we want to ~~tr~~ treat as equal.

Examples: $=$ is an equivalence relation (on any set).

"$\equiv$ mod k" is an equivalence relation $\forall k \in \mathbb{Z}$,

$\leq$ is not an equiv. rel. (it is reflexive & transitive, but not symmetric).

$\neq$ is not —— // —— (it is ~~tran~~ symmetric, but neither reflexive nor transitive),

$\sim$ for geometric shapes is an equiv. rel.,

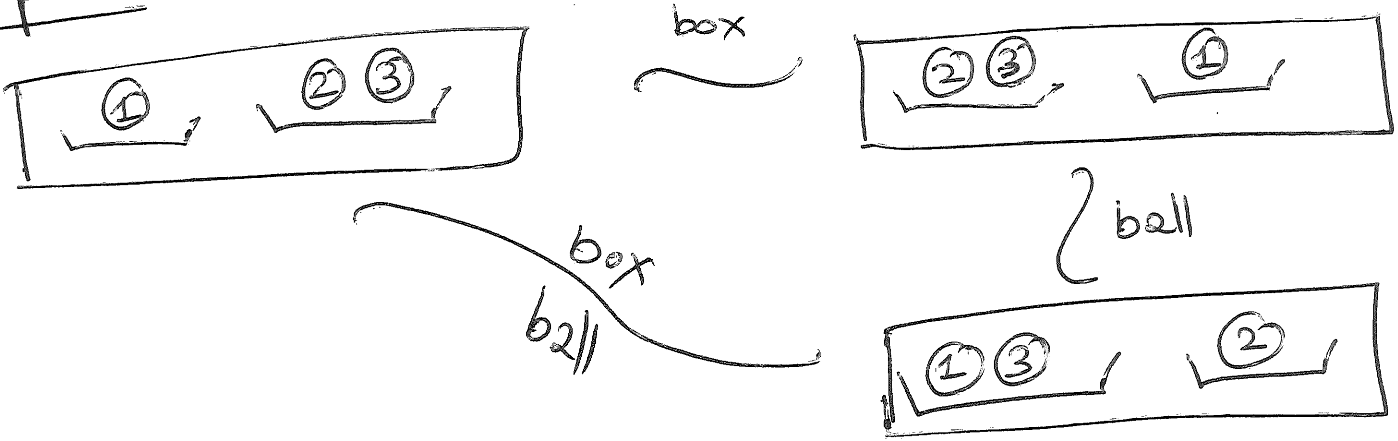
$\parallel$ for lines in the plane.

Now, go back to balls & boxes:

Def. Let $f, g : A \to X$. Then we say that

- f is <u>box-equivalent to</u> g (written $f \overset{\text{box}}{\sim} g$) if & only if $\exists$ permutation σ of X such that $f = \sigma \circ g$ (in other words, f can be obtained from g by permuting boxes).
- f is <u>ball-equivalent to</u> g (written $f \overset{\text{ball}}{\sim} g$) if & only if $\exists$ permutation τ of A such that $f = g \circ \tau$ (in other words, f can be obtained from g by permuting balls).
- f is <u>box-ball-equivalent to</u> g (written $f \underset{\text{ball}}{\overset{\text{box}}{\sim}} g$) if & only if $\exists$ permutation σ of X & a permutation τ of A such that $f = \sigma \circ g \circ \tau$.

Examples:



All of $\overset{\text{box}}{\sim}$, $\overset{\text{ball}}{\sim}$ and $\underset{\text{ball}}{\overset{\text{box}}{\sim}}$ are equivalence relations.

So counting $U \to L$ placements should mean treating ball-equivalent labellings as identical. How to do that?

Count equivalence classes.

Def. Let $\sim$ be an equivalence relation on a set S. Let $x \in S$. Then, the $\sim$-equivalence class of x, denoted by $[x]_\sim$, is defined by

$$[x]_\sim = \{y \in S \mid y \sim x\}.$$

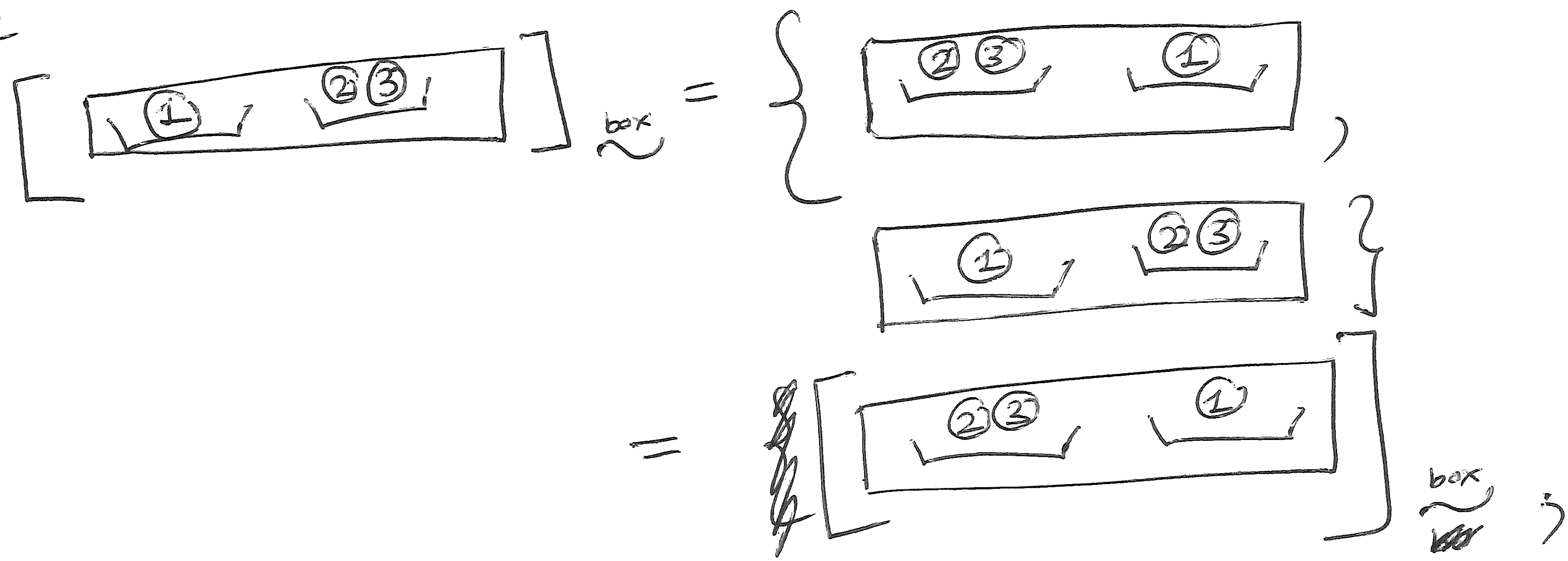
Examples: In $\mathbb{N}$, we have

$$[5]_{=} = \{y \in \mathbb{N} \mid y = 5\} = \{5\}.$$

$$[5]_{\equiv \bmod 3} = \{y \in \mathbb{N} \mid y \equiv 5 \bmod 3\} = \{2, 5, 8, 11, 14, \ldots\}.$$

$$[5]_{\equiv \bmod 2} = \{1, 3, 5, 7, \ldots\}.$$

In our running example with $|X| = 2$ and $|A| = 3$, we have



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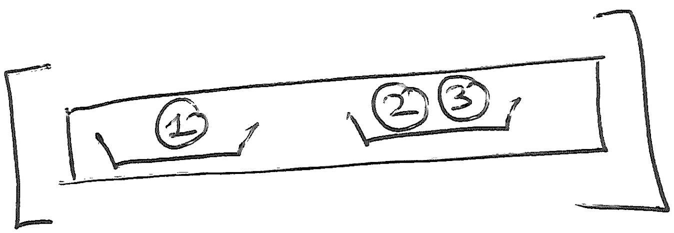
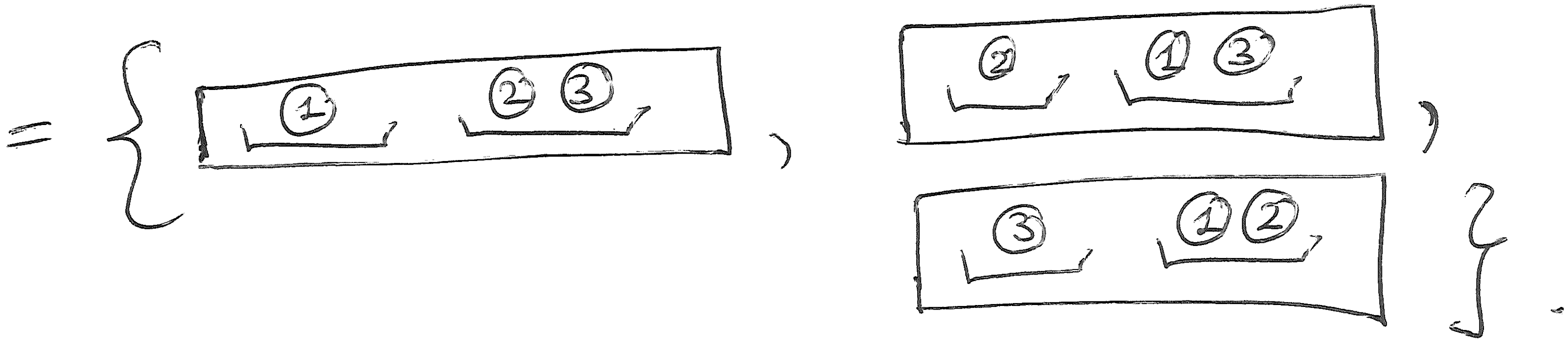


ball



Crucial fact about equivalence classes:

<u>Prop. 4.4.</u> Let $\sim$ be an equiv. rel. on a set S.

Let $x \in S$ and $y \in S$.

Then $x \sim y \iff [x]_\sim = [y]_\sim$.

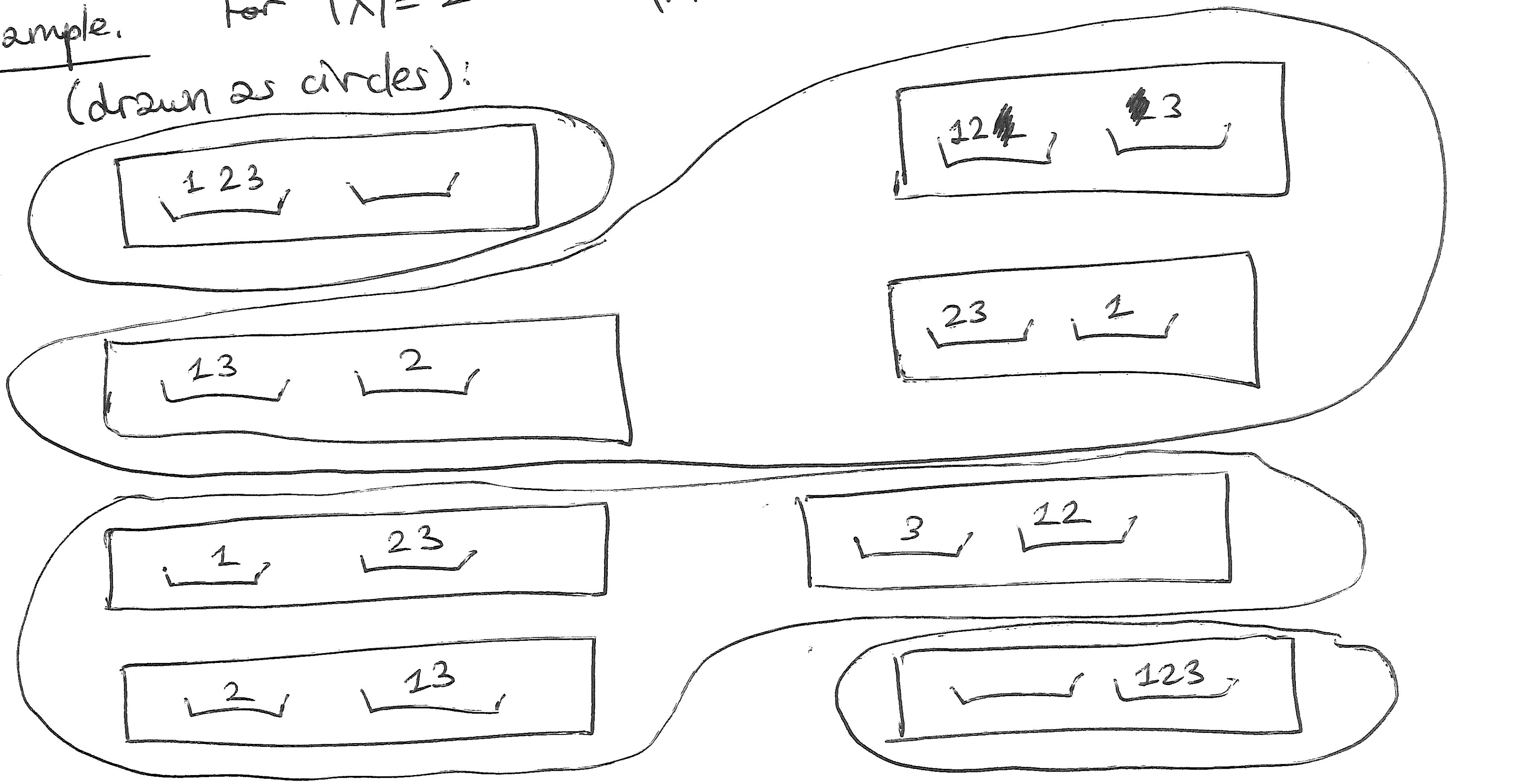
Thus, "counting elements of S up to $\sim$-equivalence"

= counting distinct $\sim$-equivalence classes.

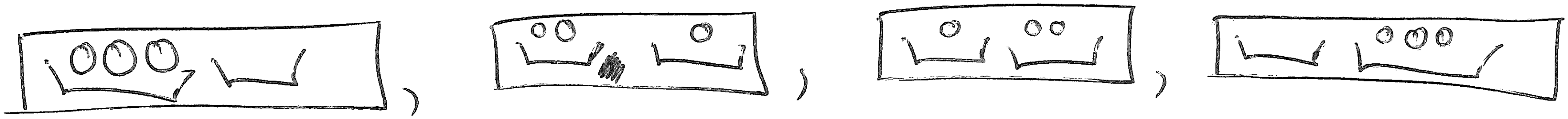
4.4. $\mathcal{U} \to L$

Def. A $\mathcal{U}\to L$ placement (i.e., placement of Unlabelled balls into Labeled boxes) is a ~~box~~ ball $\sim$-equivalence equivalence class of maps $f : A \to X$.

Example. For $|X| = 2$ and $|A| = 3$, here are the $\mathcal{U}\to L$ placements (drawn as circles):



<u>Notation:</u> When visualizing a U→L placement, we just draw the balls as circles, with no numbers in them. So the 4 U→L placements for $|X|=2$ and $|A|=3$ are



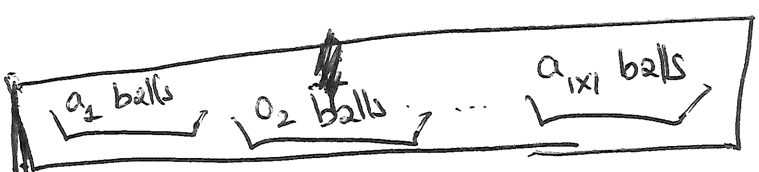
<u>Prop. 4.5.</u> (# of U→L placements $A \to X$)

$= (\#$ of $(x_1, \dots, x_{|X|}) \in \mathbb{N}^{|X|}$ satisfying $x_1 + \dots + x_{|X|} = |A|)$

$$= \binom{|A|+|X|-1}{|A|}.$$

<u>Proof.</u> <u>1st equality:</u> Consider the bijection

$\{$U→L placements$\} \to \{$weak compositions of $|A|$ into $|X|$ parts$\}$,



$\longmapsto (a_1, a_2, \dots, a_{|X|})$

(Fine print: We need to assume $X = [|X|]$.)

(We are not doing the rigorous argument.)

Thus,

(# of $U \to L$ placements)

= (# of weak compositions of $|A|$ into $|X|$ parts)

= (# of $(x_1, \ldots, x_{|X|}) \in \mathbb{N}^{|X|}$ satisfying $x_1 + \cdots + x_{|X|} = |A|$).

$\square$

2nd equality: Theorem 3.25.

Prop. 4.6. (# of surjective $U \to L$ placements)

= (# of $(x_1, \ldots, x_{|X|}) \in \{1, 2, 3, \ldots\}^{|X|}$ satisfying $x_1 + \cdots + x_{|X|} = |A|$)

$$= \begin{cases} \binom{|A|-1}{|X|-1} & \text{if } |A| \geq 1; \\ [|X| = 0] & \text{if } |A| = 0 \end{cases}$$

$$= \binom{|A|-1}{|A|-|X|}.$$

Proof. 1st equality: same argument as in Prop. 4.5.
2nd equality: Theorem 3.23.
3rd equality: symmetry of Pascal's triangle. $\square$

Prop. 4.7. (# of injective $\mathcal{U}\to L$ placements)

$= (\#$ of $(x_1,\ldots,x_{|X|}) \in \{0,1\}^{|X|}$ satisfying $x_1+\ldots+x_{|X|} = |A|)$

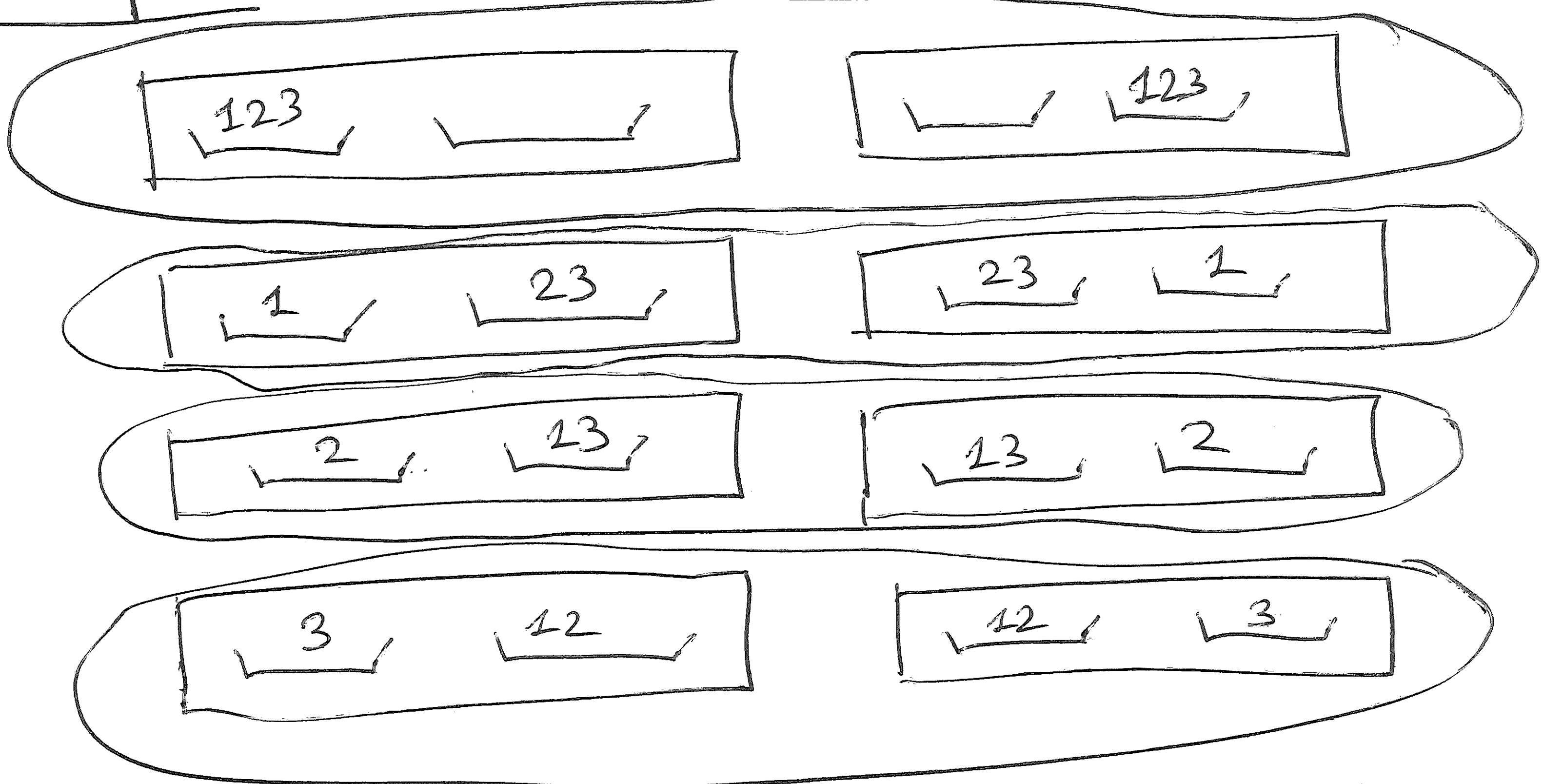
$= \binom{|X|}{|A|}$.

Proof. 1st equality: same argument as in Prop. 4.5.
2nd equality: Theorem 3.24. $\square$

4.5. $L \to \mathcal{U}$

Def. An $L\to\mathcal{U}$ placement is a $\overset{\text{box}}{\sim}$-equivalence class of maps $f: A \to X$.

Example: $|X| = 2, \ |A| = 3.$



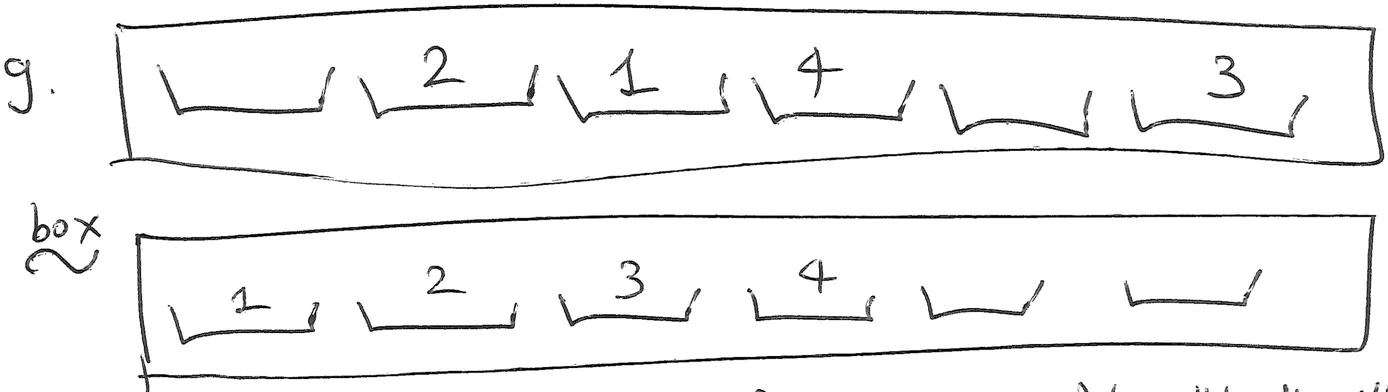
Prop. 4.8: (# of injective $L \to U$ placements)

$$= [\,|A| \leq |X|\,].$$

Proof. If $|A| > |X|$, there are no such placements (by Pigeonhole).

If $|A| \leq |X|$, then such placements exist, and are all ~~box-equivalent~~ identical:

e.g.



(since they each consist of 1 box with "ball 1", 1 box with "ball 2", …, 1 box with "ball $|A|$", and $|X|-|A|$ empty boxes),

so the # ~~of equivalence classes~~ is 1. □

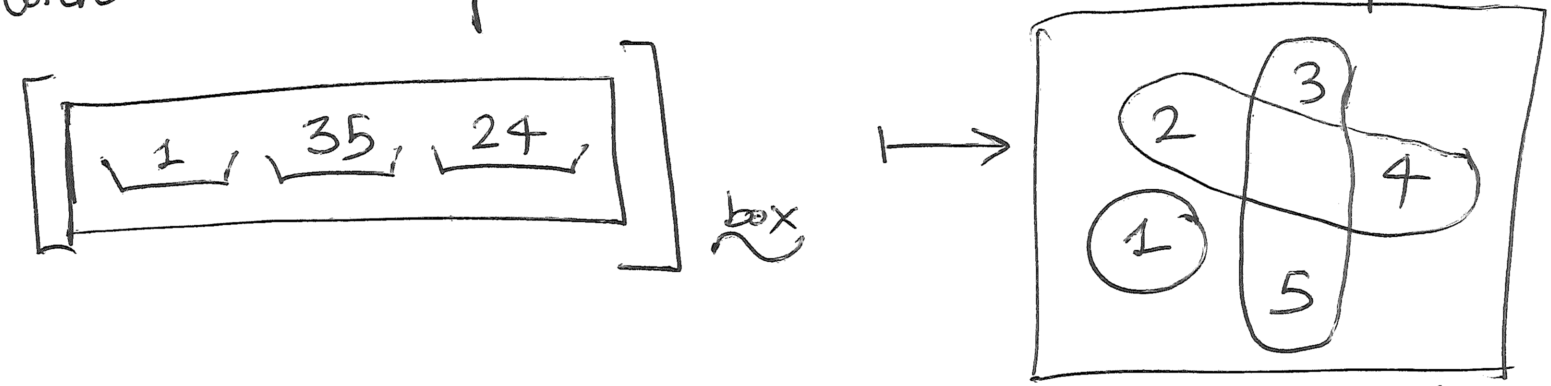
<u>Prop. 4.9.</u> (# of surjective L→U placements)

$$= \left\{ {|A| \atop |X|} \right\}$$ (a Stirling number of the 2nd kind, as defined on HW3 by $\left\{ {n \atop k} \right\} = \frac{sur(n,k)}{k!}$)

$$= \frac{sur(|A|,|X|)}{|X|!}.$$

Proof. Assume $X = [\,|X|\,]$.

Then, the surjective L→U placements are in bijection with the set partitions of A into $|X|$ parts:



$$= \{\{1\}, \{3,5\}, \{2,4\}\}.$$

So ~~the~~ (# of surjective L→U placements)

$=$ (# of set partitions of A into $|X|$ parts)

$$= \left\{ {|A| \atop |X|} \right\} \quad \text{(by some remark in HW3)}$$

$$= \frac{\operatorname{sur}(|A|, |X|)}{|X|!}.$$

$\square$

<u>Prop. 4.10.</u> (# of $L \to U$ placements)

$$= \left\{ {|A| \atop 0} \right\} + \left\{ {|A| \atop 1} \right\} + \left\{ {|A| \atop 2} \right\} + \dots + \left\{ {|A| \atop |X|} \right\}.$$

<u>Proof.</u> Exercise. $\square$