Proof of Thm. 1.12: We are in one of the following cases:

CASE 1: $k \in \{1, 2, 3, \ldots\}$.

CASE 2: $k = 0$.

CASE 3: $k \notin \mathbb{N}$.

Case 3 is easy: If $k \notin \mathbb{N}$, then $k-1 \notin \mathbb{N}$, so $\binom{n-1}{k-1} = 0$, but also $\binom{n}{k} = 0$ and $\binom{n-1}{k} = 0$, so we have to prove $0 = 0 + 0$.

Case 2: Assume $k = 0$. Then $\binom{n}{k} = \binom{n}{0} = \frac{\text{(empty product)}}{0!} = \frac{1}{1} = 1$, and similarly $\binom{n-1}{k} = 1$. Also $\binom{n-1}{k-1} = \binom{n-1}{-1} = 0$.

So we must prove $1 = 0 + 1$.

Case 1: $k \in \{1, 2, 3, \ldots\}$. Thus,

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!} + \frac{(n-1)(n-2)\cdots(n-k)}{k!}$$

$$= \frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!}\left(1 + \frac{n-k}{k}\right)$$

(since $k! = k \cdot (k-1)!$)

$$= \frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!} \cdot \frac{n}{k}$$

$$= \frac{n(n-1)\cdots\cdots(n-k+1)}{k \cdot (k-1)!} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \binom{n}{k}.$$

$\square$

<u>Thm. 1.19:</u> Let $n \in \mathbb{N}$, $k \in \mathbb{N}$ be such that $n \geq k$. Then,

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

<u>Proof.</u>

$$\binom{n}{k}(n-k)! = \frac{n(n-1)\cdots(n-k+1)}{k!}(n-k)! = \frac{n(n-1)\cdots\cdots 1}{k!} = \frac{n!}{k!}.$$

$\square$

Proof of Thm. 1.13:

CASE 1: $0 \leqslant k \leqslant n$.

CASE 2: $k < 0$.

CASE 3: $k > n$.

In Case 1, Thm 1.19 yields $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

and $\binom{n}{n-k} = \frac{n!}{(n-k)!\underbrace{(n-(n-k))}_{=k}!}$.

In Case 2: $k<0$, so $n-k>n$. Hence, Prop 1.11a ~~yields~~ (applied to $n-k$ instead of k) yields $\binom{n}{n-k} = 0$. But $k<0$ yields $\binom{n}{k} = 0$. Thus, $\binom{n}{k} = 0 = \binom{n}{n-k}$.

In Case 3: similar. $\square$

Prop. 1.20. $\forall k \in \mathbb{Z}$, we have $\binom{0}{k} = [k=0]$.

Here, we are using the Iverson bracket notation:

Def. For any statement $\mathcal{A}$, we let $[\mathcal{A}]$ be the truth value of $\mathcal{A}$, defined to be $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}$.

For example, $[1+1=2] = 1$,
$[1+1=1] = 0$,
$[\text{it's currently snowing}] = 0$,

Proof of Prop. 1.20: Three cases, as in Thm 1.13.

Proof of Thm. 1.14: We want to use induction, but we don't have an $\mathbb{N}$-variable. So let's first prove the "$n \in \mathbb{N}$" case:

Observation 1: $\binom{n}{k} \in \mathbb{N}$ whenever $n \in \mathbb{N}$, $k \in \mathbb{Z}$.

Proof of Observation 1: Induction on n.

Base case: $\binom{0}{k} = [k=0] \in \{0,1\} \subseteq \mathbb{N}$.

<u>Step:</u> Let $m \in \mathbb{N}$. Assume (as the IH = induction hypothesis) that Observation 1 holds for $n = m$. We must prove that Observation 1 holds for $n = m+1$.

Let $k \in \mathbb{Z}$. Then, Thm 1.12 says

$$\binom{m+1}{k} = \binom{(m+1)-1}{k-1} + \binom{(m+1)-1}{k}$$

$$= \underbrace{\binom{m}{k-1}}_{\substack{\in \mathbb{N} \\ \text{(by IH)}}} + \underbrace{\binom{m}{k}}_{\substack{\in \mathbb{N} \\ \text{(by IH)}}} \in \mathbb{N}.$$

Thus, Observation 1 holds for $n = m+1$.

So Observation 1 is proven.

Now, prove Thm. 1.14:

CASE 1: $n \geq 0$.

CASE 2: $n < 0$.

In Case 1: $n \in \mathbb{N}$, so Observation 1 yields

$$\binom{n}{k} \in \mathbb{N} \subseteq \mathbb{Z}.$$

In Case 2: $n < 0$. So $n \leq -1$.

Prop. 1.11b (applied to $-n$ instead of n) yields

$$\binom{n}{k} = (-1)^k \binom{-n+k-1}{k}.$$

Thus, if $-n+k-1 \in \mathbb{N}$, then Observation 1 yields

$$\binom{n}{k} = (-1)^k \underbrace{\binom{-n+k-1}{k}}_{\in \mathbb{N}} \in \mathbb{Z},$$

What if $-n+k-1 \notin \mathbb{N}$? Then, $-n+k-1 < 0$.

Add $n \leq -1$, obtain $k-1 < -1$, so $k < 0$,

So $\binom{n}{k} = 0 \in \mathbb{Z}$. $\square$

<u>Proof of Thm. 1.15.</u> Induction on n.

<u>Base:</u> If S is a 0-elt. set, then

$$\begin{aligned} & \# \text{ of } k\text{-elt. subsets of } S \\ = \; & \# \text{ of } k\text{-elt. subsets of } \varnothing \\ = \; & [k=0] = \binom{0}{k} \qquad \text{(by Prop. 1.20)}. \end{aligned}$$

<u>Step:</u> Let $m \in \mathbb{N}$. Assume (as IH) that Thm. 1.15 holds for $n=m$. We must show it holds for $n=m+1$.

Let S be an $(m+1)$-elt. set.

Thus, S is nonempty. So $\exists\, t \in S$. Fix such a t.

Now, there are two ~~kinds of subs~~ types of subsets of S:

<u>Type 1:</u> those that contain t.

<u>Type 2:</u> those that don't contain t.

So

$$\begin{aligned}
&\#\text{ of }k\text{-elt. subsets of }S\\
&=(\#\text{ of }k\text{-elt. subsets of }S\text{ of Type 1})\\
&\quad+(\#\text{ of }k\text{-elt. subsets of }S\text{ of Type 2}).
\end{aligned} \tag{1}$$

But the subsets of S of Type 2 are the subsets of $S\setminus\{t\}$. So

$$\begin{aligned}
&(\#\text{ of }k\text{-elt. subsets of }S\text{ of Type 2})\\
&=(\#\text{ of }k\text{-elt. subsets of }\underbrace{S\setminus\{t\}}_{\text{an }m\text{-elt. set}}) = \binom{m}{k}.
\end{aligned}$$

What about Type 1?

Informally: The subsets of S of Type 1 with k elements "correspond to" the subsets of $S\setminus\{t\}$ with $k-1$ elements.

Formally: The map

~~{Subsets~~

$$\{k\text{-elt. subsets of } S \text{ of Type } 1\} \to \{(k-1)\text{-elt. subsets of } S\setminus\{t\}\},$$
$$Q \mapsto Q\setminus\{t\}$$

is a bijection (its inverse is

$$\{(k-1)\text{-elt. subsets of } S\setminus\{t\}\} \to \{k\text{-elt. subsets of } S \text{ of Type } 1\},$$
$$R \mapsto R\cup\{t\}\ ).$$

But if X and Y are finite sets and if $\exists$ bijection from X to Y, then $|X| = |Y|$.

Thus, $|\{k\text{-elt. subsets of } S \text{ of Type } 1\}|$
$= |\{(k-1)\text{-elt. subsets of } S\setminus\{t\}\}|$.

In other words,

$$(\#\text{ of } k\text{-elt. subsets of } S \text{ of Type } 1)$$
$$= (\#\text{ of } (k-1)\text{-elt. subsets of } \underbrace{S\setminus\{t\}}_{m\text{-elt. set}}) \underset{\text{IH}}{=\!=} \binom{m}{k-1}.$$

So (1) becomes

of k-elt. subsets of S

$$= \binom{m}{k-1} + \binom{m}{k} \underset{\substack{\text{Thm 1.12} \\ \text{(applied to } n=m+1)}}{=\!=\!=} \binom{m+1}{k}.$$

Thus, Thm 1.15 holds for $n = m+1$. This completes the step. $\square$

Proof of Thm. 16.

I will rewrite $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ as $\sum_{k \in \mathbb{Z}} \binom{n}{k} x^k y^{n-k}$

(that is, a sum over all integers k).

Why is this infinite sum well-defined?

Because only finitely many of its addends are $\neq 0$.

For example, if $n=3$, it has the form

$$\cdots + 0 + 0 + 0 + x^3 + 3x^2y + 3xy^2 + y^3 + 0 + 0 + 0 + \cdots$$

An infinite sum that has only finitely many nonzero terms is always well-defined. Its value is obtained by discarding the 0's.

Since $\binom{n}{k} = 0$ whenever $k \notin \{0, 1, \ldots, n\}$, we ~~thus~~ thus see that $\sum_{k \in \mathbb{Z}} \binom{n}{k} x^k y^{n-k}$ is well-defined & equals

$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

So ~~we~~ it remains to prove

(2) $$(x+y)^n = \sum_{k \in \mathbb{Z}} \binom{n}{k} x^k y^{n-k}.$$

Now we'll prove this by induction:

<u>Base</u>: easy.

<u>Step</u>: Assume (2) holds for $n = m$ (where $m \in \mathbb{N}$ is fixed).
Must prove (2) holds for $n = m+1$.

We have

$$(x+y)^{m+1} = \underbrace{(x+y)^m}_{\substack{=\sum_{k\in\mathbb{Z}}\binom{m}{k}x^k y^{m-k} \\ \text{(by IH)}}}(x+y)$$

$$= \left(\sum_{k\in\mathbb{Z}}\binom{m}{k}x^k y^{m-k}\right)(x+y)$$

$$= \left(\sum_{k\in\mathbb{Z}}\binom{m}{k}x^k y^{m-k}\right)\cdot x + \left(\sum_{k\in\mathbb{Z}}\binom{m}{k}x^k y^{m-k}\right)\cdot y$$

$$= \sum_{k\in\mathbb{Z}}\binom{m}{k}x^{k+1}y^{m-k} + \sum_{k\in\mathbb{Z}}\binom{m}{k}x^k y^{m-k+1}$$

$$= \sum_{k \in \mathbb{Z}} \binom{m}{k-1} x^{(k-1)+1} y^{m-(k-1)} + \sum_{k \in \mathbb{Z}} \binom{m}{k} x^k y^{m-k+1}$$

(here, we substituted $k-1$ for k in the 1st sum)

$$= \sum_{k \in \mathbb{Z}} \binom{m}{k-1} x^k y^{m+1-k} + \sum_{k \in \mathbb{Z}} \binom{m}{k} x^k y^{m+1-k}$$

$$= \sum_{k \in \mathbb{Z}} \underbrace{\left(\binom{m}{k-1} + \binom{m}{k} \right)}_{= \binom{m+1}{k}} x^k y^{m+1-k}$$

$$= \sum_{k \in \mathbb{Z}} \binom{m+1}{k} x^k y^{m+1-k}.$$

Thus, (2) holds for $n=m+1$. $\square$

(See [detnotes, §2.1] for a proof without "infinite sums".)

The proof of ~~Prop.~~ Thm. 1.16 is an exercise in induction.
The proof of Prop. 1.17 is an exercise on HW1.

1.4. Counting

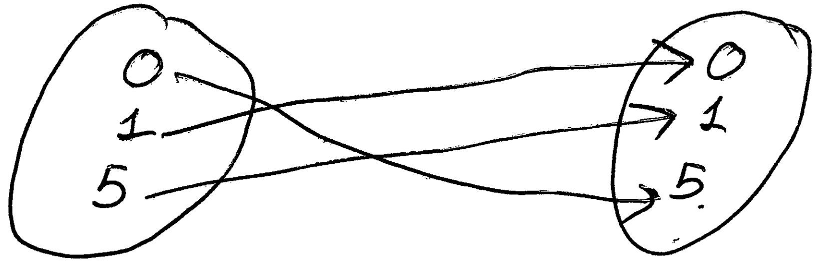
Counting := enumeration := finding sizes of finite sets.

E.g., what we have done in Thm. 1.6 & Thm. 1.15.

Much more can be done. Some examples:

Def. A *permutation* of a set X is a bijection $X \to X$.

For example,



is a permutation of $\{0,1,5\}$.

Thm 1.21. Let $n \in \mathbb{N}$. Let X be an n-elt. set. Then, (# of permutations of X) $= n!$.

(Proof will be given later.)

Def. A <u>derangement</u> of a set X means a permutation σ of X such that $\sigma(x) \neq x$ $\forall x \in X$.

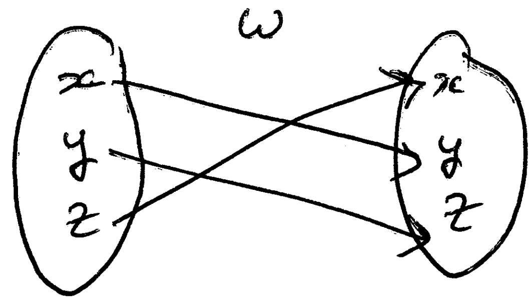
How many derangements does an n-elt. set have?

Def. For each $n \in \mathbb{N}$, let $[n]$ be the set $\{1, 2, \ldots, n\}$.

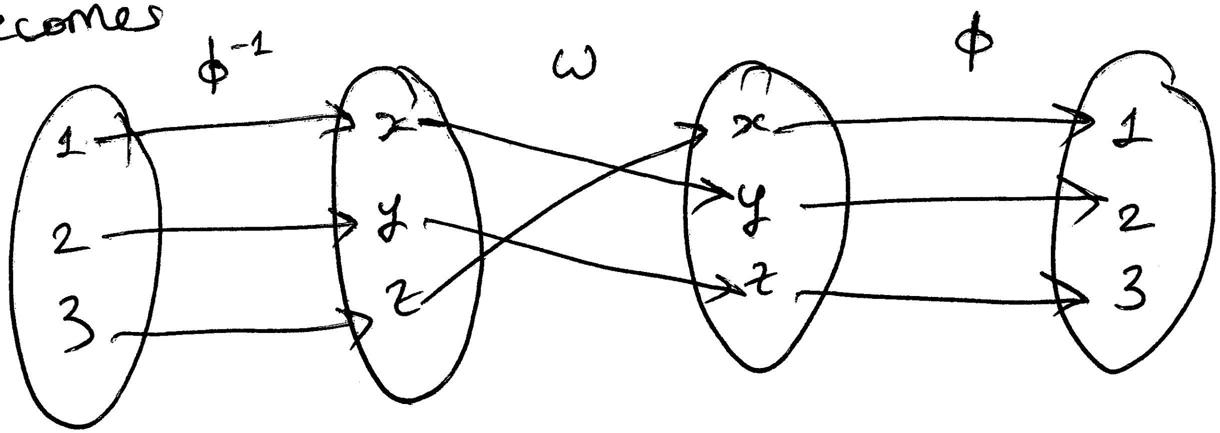
Instead of studying an arbitrary n-elt. set X, it suffices to study $[n]$:

Lem. 1.22. Let X be any n-elt. set. Then, (# of derangements of X) $=$ (# of derangements of $[n]$).

Proof. Fix a bijection $\phi : X \to [n]$. (It exists, since X has n elts.) Now, any derangement of X can be transformed into a derangement of $[n]$ by "relabeling" the elements of X as $1, 2, \ldots, n$ using ϕ. For example, the derangement



becomes



$\phi \circ \omega \circ \phi^{-1}$

So, formally,

$$\{\text{derangements of } X\} \longrightarrow \{\text{derangements of } [n]\},$$
$$\omega \longmapsto \phi \circ \omega \circ \phi^{-1}$$

is a bijection (its inverse being

$$\{\text{derangements of } [n]\} \longrightarrow \{\text{derangements of } X\},$$
$$\alpha \longmapsto \phi^{-1} \circ \alpha \circ \phi \;).$$

Thus, $\#(\text{derangements of } X) = \#(\text{derangements of } [n])$, $\square$

So it suffices to count derangements of $[n]$.

__Def.__ For each $n \in \mathbb{N}$, let $D_n = \#(\text{derangements of } [n])$.

~~Ex~~ __Def.__ Let $n \in \mathbb{N}$. The __one-line notation__ of a permutation σ of $[n]$ is the n-tuple $(\sigma(1), \sigma(2), \dots, \sigma(n))$.

__Ex:__ The permutations of $[3]$ are

$(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$,

$(3,1,2)$, $(3,2,1)$.

Only $(2,3,1)$ and $(3,1,2)$ are derangements of $[3]$.
Thus $D_3 = 2$.

Ex: $D_0 = 1$ (since $\mathrm{id}: \emptyset \to \emptyset$ is a derangement),

$D_1 = 0$ (since $\mathrm{id}: [1] \to [1]$ is NOT a derangement),

$D_2 = 1$,

$D_3 = 2$,

$D_4 = 9$.

Thm. 1.20. (a) $D_n = (n-1)\left(D_{n-1} + D_{n-2}\right) \quad \forall n \geq 2$.

(b) $D_n = n D_{n-1} + (-1)^n \quad \forall n \geq 1$.

(c) $$n! = \sum_{k=0}^{n} \binom{n}{k} D_k \quad \forall n \in \mathbb{N}.$$

(d) $$D_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}.$$

(e) $$D_n = \mathrm{round}\left(\frac{n!}{e}\right) = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor \quad \forall n \geq 1.$$ (where $e \approx 2.718\ldots$).