1.2. Sums of powers

Thm. 1.8. ("Little Gauss"). Let $n \in \mathbb{N}$. (Keep in mind: $0 \in \mathbb{N}$.)

Then, $1 + 2 + \dots + n = \frac{n(n+1)}{2}$.

Rmk. If $n = 0$, then $1 + 2 + \dots + n = 1 + 2 + \dots + 0 = (\text{empty sum}) = 0$.

1st proof:

$$\begin{aligned} 2(1 + 2 + \dots + n) &= (1 + 2 + \dots + n) \\ &\quad + (1 + 2 + \dots + n) \\ &= (1 + 2 + \dots + n) \\ &\quad + (n + (n-1) + \dots + 1) \\ &= \underbrace{(n+1) + (n+1) + \dots + (n+1)}_{n \text{ times}} = n(n+1). \end{aligned}$$

$\square$

2nd proof. Induction over n.

Base case: Thm. 1.8 for $n = 0$ is true: $0 = 0$.

Step: Fix $m \in \mathbb{N}$. Assume that Thm. 1.8 is true for $n = m$.
Want to prove that Thm 1.8 is true for $n = m + 1$.

By assumption, $1+2+\dots+m = \frac{m(m+1)}{2}$.

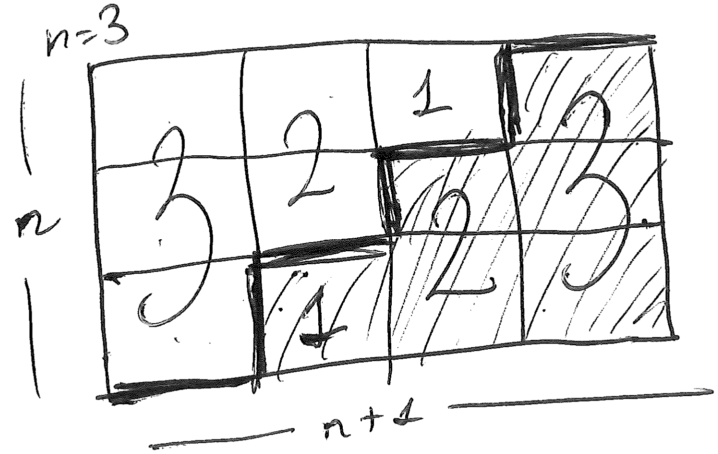
We need to prove $1+2+\dots+(m+1) = \frac{(m+1)((m+1)+1)}{2}$.

But this follows from

$$1+2+\dots+(m+1) = \underbrace{(1+2+\dots+m)}_{=\frac{m(m+1)}{2}} + (m+1)$$

$$= \frac{m(m+1)}{2} + (m+1) = \frac{(m+1)((m+1)+1)}{2}.$$ □

3rd proof (outline). Idea:



$$3 \cdot 4 = (1+2+3) + (1+2+3)$$

□

Rmk. Why is $1+2+\dots+n$ well-defined?

E.g. why is $(\cdots((1+2)+3)+\cdots)+n$

$$= 1 + (\cdots + ((n-2) + ((n-1)+n))\cdots) \quad ?$$

We'll see this later, it's proven using induction.
("General commutativity").

Prop. 1.9. Let $n \in \mathbb{N}$. Then, $1^2+2^2+\dots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Proof. By induction, just as the 2nd proof of Thm. 1.8.

Prop. 1.10. Let $n \in \mathbb{N}$. Then, $1^3+2^3+\dots+n^3 = \dfrac{n^2(n+1)^2}{4}$.

Proof. By induction.

Question: Given $k \in \mathbb{N}$, what is $1^k+2^k+\dots+n^k$?
A $(k+1)$-th degree polynomial in n?

Def. If $a_p, a_{p+1}, \dots, a_q$ (for some integers p, q) are numbers (or matrices), then $\sum_{i=p}^{q} a_i$ means $a_p + a_{p+1} + \dots + a_q$.

If $p > q$, then we understand this to mean 0.

So $1^k + 2^k + \dots + n^k = \sum_{i=1}^{n} i^k$ ← (not the i from complex analysis)

Thm. 1.11. Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Assume $k > 0$.

Then, $$1^k + 2^k + \dots + n^k = \sum_{i=0}^{k} \operatorname{sur}(k,i) \cdot \binom{n+1}{i+1},$$

where:

- $\operatorname{sur}(k,i) :=$ # of surjective (= onto) maps $\{1,2,\dots,k\} \to \{1,2,\dots,i\}$. (This is $\emptyset$ if $i=0$)

- $$\binom{x}{j} := \frac{x(x-1)\cdots(x-j+1)}{j(j-1)\cdots 1} \quad \forall x \in \mathbb{R},\ j \in \mathbb{N}.$$

(We won't prove this right now.)

Example: Let $k=2$. Then, Thm. 1.11 yields

$$1^2 + 2^2 + \dots + n^2 = \sum_{i=0}^{2} \operatorname{sur}(2,i) \cdot \binom{n+1}{i+1}$$

$$= \underbrace{\operatorname{sur}(2,0)}_{=0} \binom{n+1}{1} + \underbrace{\operatorname{sur}(2,1)}_{=1} \binom{n+1}{2} + \underbrace{\operatorname{sur}(2,2)}_{=2} \binom{n+1}{3}$$

$$= \cancel{0\binom{n+1}{1}} + 1\underbrace{\binom{n+1}{2}}_{=\frac{(n+1)n}{2\cdot 1}} + 2\underbrace{\binom{n+1}{3}}_{=\frac{(n+1)n(n-1)}{3\cdot 2\cdot 1}}$$

$$= \frac{(n+1)n}{2\cdot 1} + 2\frac{(n+1)n(n-1)}{3\cdot 2\cdot 1} = \frac{(n+1)n(3+2(n-1))}{6}$$

$$= \frac{n(n+1)(2n+1)}{6},$$

so Prop 1.9 follows.

Similarly we can get Thm 1.8 & Prop. 1.10.

(Thm. 1.11 is [Galvin, Prop. 23.2].)

1.3. Factorials & binomial coefficients

Def. For any $n \in \mathbb{N}$, set $n! = 1 \cdot 2 \cdot \cdots \cdot n$.

This is understood to be 1 if $n = 0$, since generally, empty products are understood to be 1.

We refer to $n!$ as "n factorial".

Rmk. If n is a pos. int., then

$$n! = 1\cdot 2\cdot \ldots \cdot n = \underbrace{(1\cdot 2\cdot \ldots \cdot (n-1))}_{=(n-1)!}\cdot n$$

$$= (n-1)!\cdot n.$$

Ex: $0! = 1$, $1! = 1$, $2! = 1\cdot 2 = 2$, $3! = 1\cdot 2\cdot 3 = 6$,
$4! = 24$, $5! = 120$, $6! = 720$, $7! = 5040$,
$8! = 40\,320$, ...

Def. Let n be a number ($n \in \mathbb{N}$, $n \in \mathbb{Z}$, $n \in \mathbb{R}$, $n \in \mathbb{C}$).

(a) For any $k \in \mathbb{N}$, set

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

(b) If $k \notin \mathbb{N}$, set $\binom{n}{k} = 0$.
(This is the convention of [GrKnPa].)

We call $\binom{n}{k}$ a binomial coefficient, and we refer to it as "n choose k".

Ex: $\binom{n}{0} = \frac{\text{(empty product)}}{0!} = \frac{1}{1} = 1.$

$\binom{n}{1} = \frac{n}{1!} = n.$

$\binom{n}{2} = \frac{n(n-1)}{2!} = \frac{n(n-1)}{2}.$

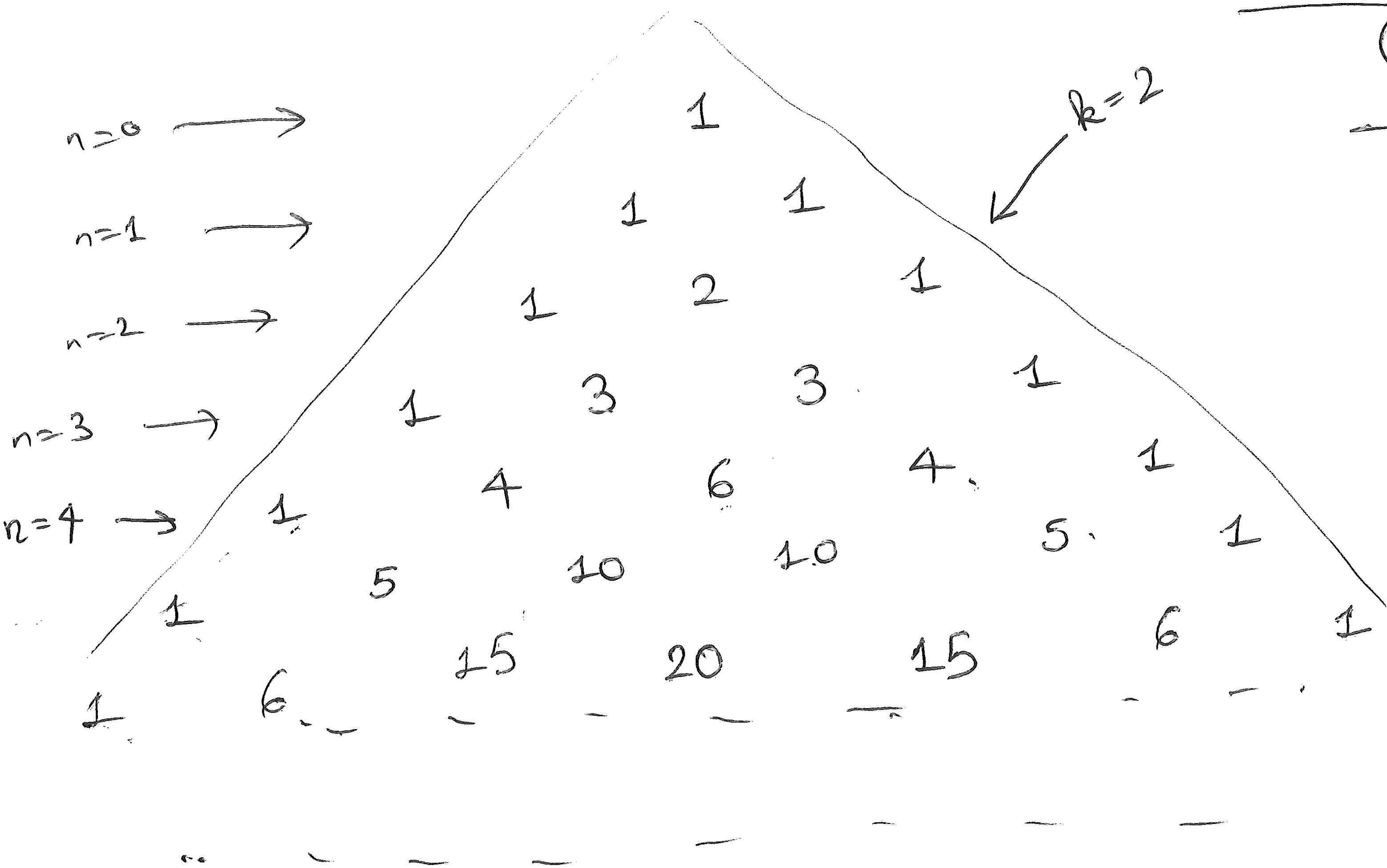
$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!} = \frac{n(n-1)(n-2)}{6}.$

$\binom{-1}{5} = \frac{(-1)(-2)(-3)(-4)(-5)}{5!} = \frac{(-1)^5\, 5!}{5!} = (-1)^5.$

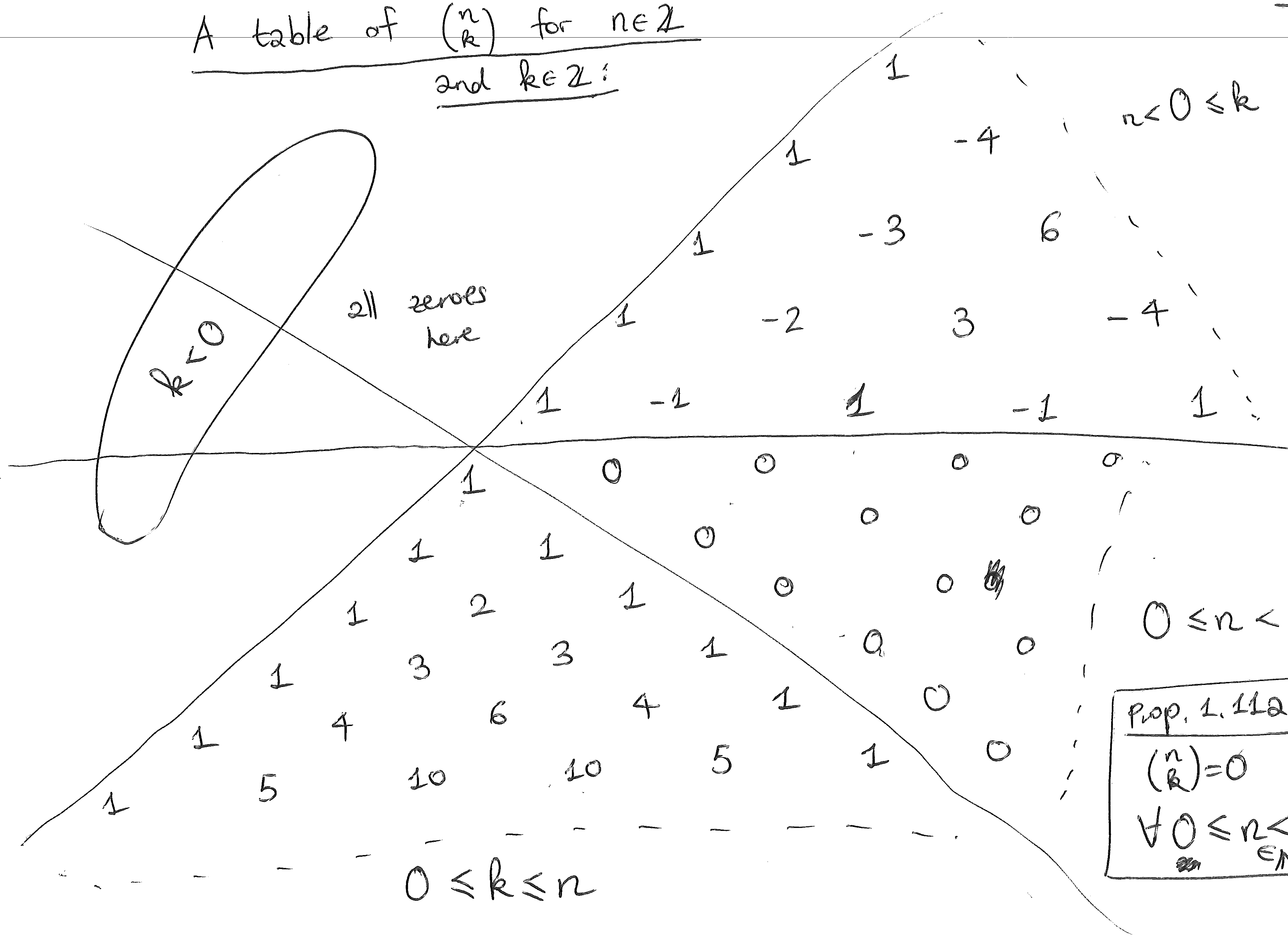
$\binom{\sqrt{2}}{2} = \frac{\sqrt{2}(\sqrt{2}-1)}{2} = \frac{2-\sqrt{2}}{2}.$

$\binom{2}{\sqrt{2}} = 0$ since $\sqrt{2} \notin \mathbb{N}$.

"Pascal's triangle": a table of $\binom{n}{k}$ for $0 \leq k \leq n$ (and $k, n \in \mathbb{N}$).



A table of $\binom{n}{k}$ for $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$:



Prop. 1.11a.

$\binom{n}{k} = 0 \quad \forall\, 0 \leq n < k \in \mathbb{N}.$

Prop. 1.11a. If $n \in \mathbb{N}$ and $k > n$, then $\binom{n}{k} = 0$.

Proof.
$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{0}{k!},$$
since 0 is one of the factors $n, n-1, \ldots, n-k+1$. $\square$

Prop. 1.11b ("upper negation"). Let $n \in \mathbb{R}$ and $k \in \mathbb{N}$.

Then,
$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

Proof.
$$\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} = \frac{(-1)^k n(n+1)\cdots(n+k-1)}{k!}$$
$$= \frac{(-1)^k (n+k-1)(n+k-2)\cdots n}{k!} = (-1)^k \underbrace{\frac{(n+k-1)(n+k-2)\cdots n}{k!}}_{=\binom{n+k-1}{k}}$$
$$= (-1)^k \binom{n+k-1}{k}.$$
$\square$

Here are some more results, to be proven later:

Thm. 1.12. (Recurrence of BC).

Let $n \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Thm. 1.13. (Symmetry of BC).

Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then, $\binom{n}{k} = \binom{n}{n-k}$.

Thm. 1.14. (Integrality of BC).

Let $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Then, $\binom{n}{k} \in \mathbb{Z}$.

Thm. 1.15. (Comb. interpr. of BC)

Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Let S be an n-elt. (element) set.

Then, $\binom{n}{k} = \#$ of k-elt. subsets of S.

Ex: Let $n = 4$ and $k = 2$ and $S = \{1, 2, 3, 4\}$.
Then, the 2-elt. subsets of S are
$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$.

Their number is $6 = \binom{4}{2}$, as Thm 1.15 says.

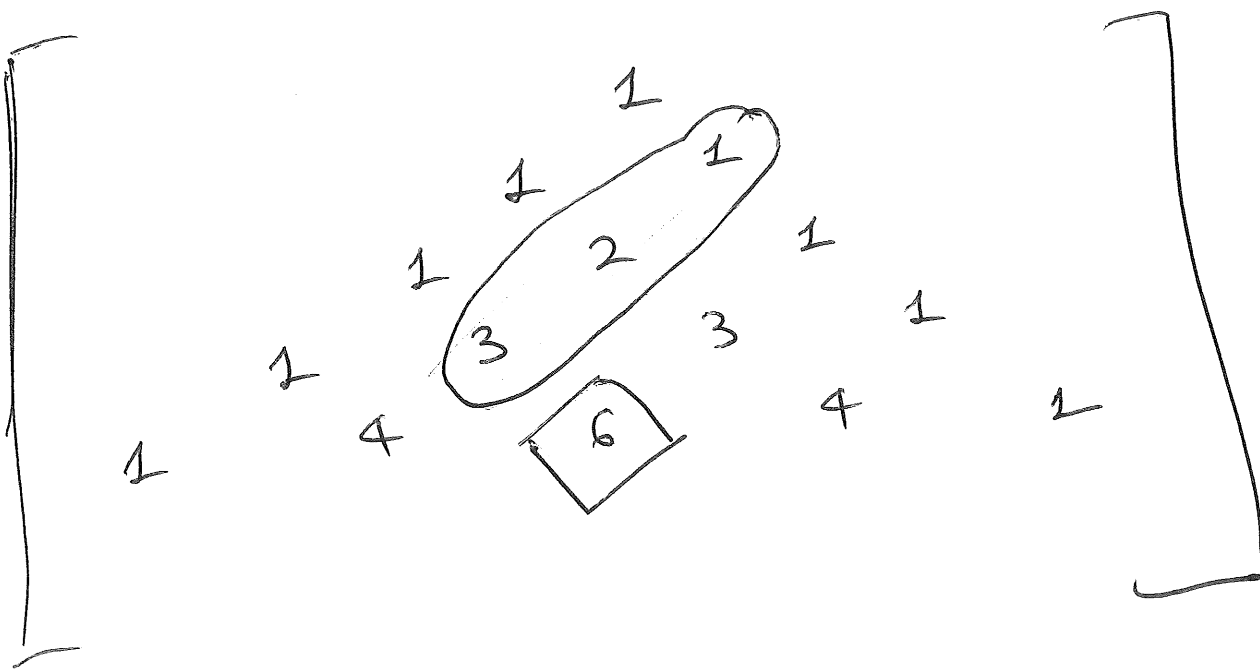
We'll prove these later. (See also [detnotes, Ch. 2].)

<u>Warning:</u> Thm. 1.15 says nothing about $\binom{n}{k}$ when $n \notin \mathbb{N}$. Thm. 1.13 does not hold if n is negative.

<u>Thm. 1.16. ("Hockey-stick identity").</u>

Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then,

$$\binom{0}{k} + \binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$



Note that if $k = 1$, this becomes

$$0 + 1 + \cdots + n = \binom{n+1}{2} = \frac{(n+1)n}{2}.$$

This is Thm. 1.8.

Thm. 1.16a (binomial formula). Let $x, y \in \mathbb{R}$. Let $n \in \mathbb{N}$. Then, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Cor. 1.16b. Let $n \in \mathbb{N}$. Then, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof. Set $x=1$ & $y=1$ in Thm. 1.16a.

Prop. 1.17. Let $n \in \mathbb{N}$. Let $a, b \in \{0, 1, \ldots, 2^n - 1\}$.

Then:

(a) $\binom{2^n + a}{b} \equiv \binom{a}{b} \mod 2$.

[Rmk: If u, v, k are three integers, then $u \equiv v \mod k$ means $k \mid u - v$.

So $u \equiv v \mod 2$ means that u, v are both even or both odd.]

(b) $\binom{2^n + a}{2^n + b} \equiv \binom{a}{b} \mod 2$.

Prop. 1.18. Let $n \in \mathbb{N}$. Then, the Fibonacci number f_{n+1} is

$$f_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k} = \binom{n-0}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \dots + \binom{n-n}{n}.$$

~~1.4. Counting~~

For a proof of Prop. 1.18, see Math 4707 Fall 2017 HW2 solution to Exercise 3 (b). Or you can show it by strong induction (see later).