http://www.math.~~edu~~umn.edu/~dgrinber/comb2/

4707 Lec 1.

Office hrs: Mon 16:35 – 17:35,
Tue 14:00 – 15:00,
Wed 16:35 – 17:35.

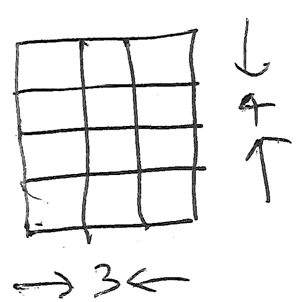
1. Introduction

Let's give some simple / concrete examples of combinatorics (mostly, enumerative combinatorics).

1.1. Domino tilings

~~Give~~ Let $R_{n,m}$ be a rectangle with width n & height m. ($n, m \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.)

Ex: $R_{3,4}$ =



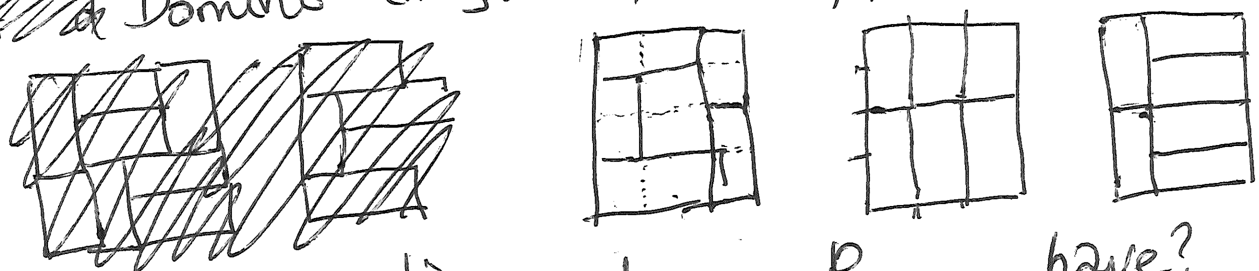
A *domino* is a 1×2-rectangle or a 2×1-rectangle

(horizontal)

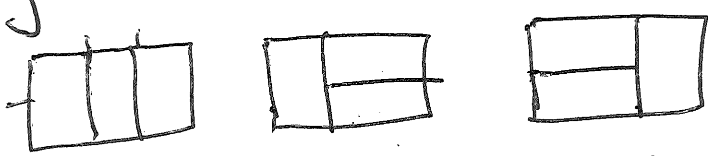
(vertical)

A *domino tiling* of $R_{n,m}$ is a way to subdivide $R_{n,m}$ into non-intersecting dominos.

E.g.: Domino tilings of $R_{3,4}$:



How many domino tilings does $R_{n,m}$ have?

$R_{3,2}$:  $\Rightarrow$ 3

How to define domino tilings rigorously?

Geometrically: we'd have to make sense of "subdivide". This is hard.

Combinatorially: Redefine $R_{n,m}$ as the set $[n] \times [m]$, where $[k] = \{1, 2, \ldots, k\}$ for all $k \in \mathbb{N}$.

Its elements are the pairs (i,j) with $i \in [n]$ and $j \in [m]$. So $|R_{n,m}| = nm$.

A <u>vertical domino</u> is a set of the form $\{(i,j), (i,j+1)\}$ for some $i, j \in \mathbb{Z}$.

A <u>horizontal domino</u> is a set of the form $\{(i,j), (i+1,j)\}$ for some $i,j \in \mathbb{Z}$.

A <u>domino tiling</u> of a set S is a set of dominos (i.e., vertical or horizontal dominos) whose union is S and which are disjoint.

Let $d_{n,m}$ = # of all domino tilings of $R_{n,m}$.

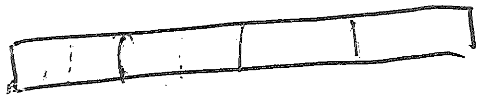
<u>Problem:</u> $d_{n,m} = ?$

<u>Prop. 1.1:</u> If n and m are odd, then $d_{n,m} = 0$.

<u>Proof.</u> Assume n and m are odd $\Rightarrow$ $|R_{n,m}| = nm$ is odd.
But each domino has even size.
If there was a domino tiling, the sum of the sizes of the dominos would be even, but it would be $|R_{n,m}| = nm$, which is odd. ↯ $\rightarrow\leftarrow$

Prop. 1.2. If $m=1$ and n is even, then $d_{n,m}=1$.

Proof.



Next case: $m=2$.

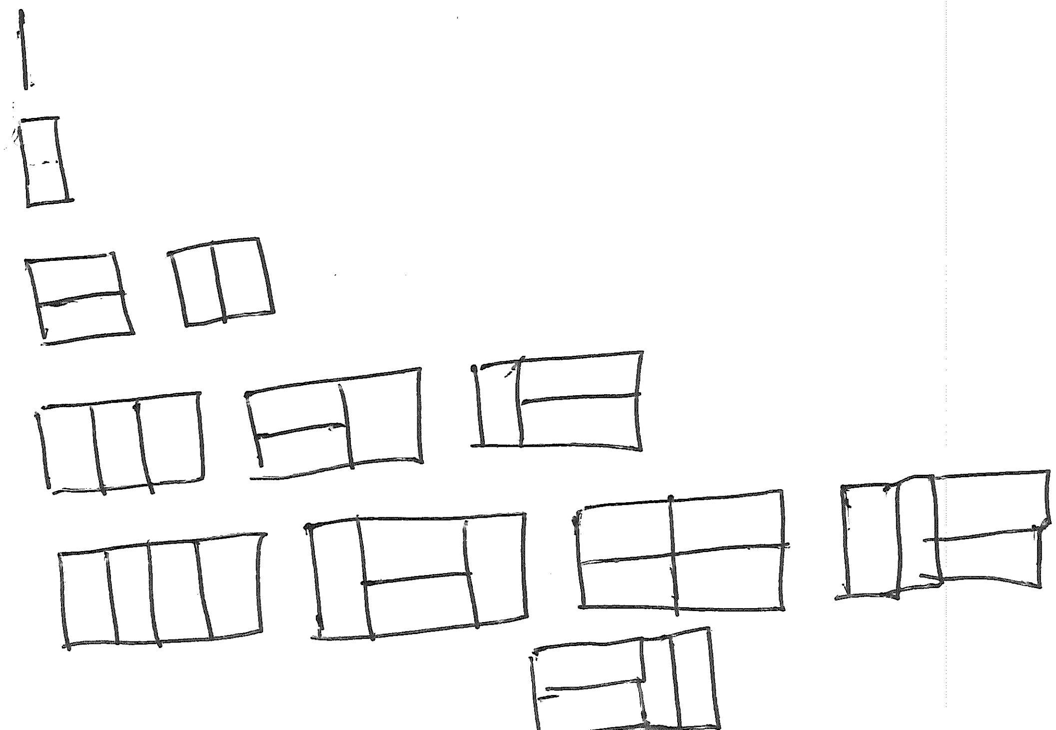
$d_{0,2}=1$

$d_{1,2}=1$

$d_{2,2}=2$

$d_{3,2}=3$

$d_{4,2}=5$



$d_{5,2}=?$

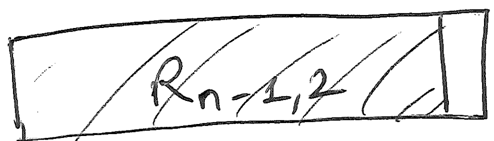
Prop. 1.3. For each $n \geq 2$, we have $d_{n,2}=d_{n-1,2}+d_{n-2,2}$.

Proof. Given a domino tiling T of $R_{n,2}$, consider the last column (i.e., $\{(n,1),(n,2)\}$).

This column is <u>either</u> covered by one vertical domino,

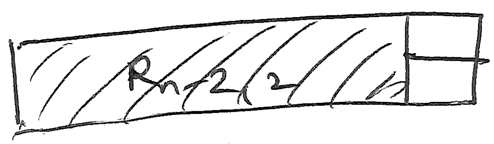
<u>or</u> covered by two horizontal dominos.

In the former case, T consists of the vertical domino covering the last column and a domino tiling of $R_{n-1,2}$



There are $d_{n-1,2}$ such tilings.

In the latter case, the two horizontal dominos cover precisely the last 2 columns, and the rest of T covers the first $n-2$ columns, so it is a domino tiling of $R_{n-2,2}$.



There are $d_{n-2,2}$ such tilings.

So the total # of T's is $d_{n-1,2} + d_{n-2,2}$.

But this # is $d_{n,2}$.

$\Rightarrow d_{n,2} = d_{n-1,2} + d_{n-2,2}$. $\square$

Ex: $d_{5,2} = d_{4,2} + d_{3,2} = 5+3=8$.

$d_{6,2} = d_{5,2} + d_{4,2} = 8+5=13$.

Def. The Fibonacci sequence is a sequence $(f_0, f_1, f_2, \dots)$ of nonnegative integers defined recursively by

$f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ $\forall n \geq 2$.

Ex:

n	0	1	2	3	4	5	6	7	8	9	......
f_n	0	1	1	2	3	5	8	13	21	34	

Prop. 1.4: $d_{n,2} = f_{n+1}$ $\forall n \in \mathbb{N}$.

Proof. Strong induction over n:

Base: True for $n=0$ and for $n=1$. (Check it.)

Step: Fix $N \in \mathbb{N}$.

Assume Prop. 1.4 is true for all $n < N$.
This is the ind. hypothesis.

We must show: Prop. 1.4 is true for $n = N$.
In other words: we must show $d_{N,2} = f_{N+1}$.

Assume $N \geq 2$ (else this follows from Base).
So Prop. 1.3. yields

$$d_{N,2} = \underbrace{d_{N-1,2}}_{\substack{=f_N \\ \text{(by ind. hyp.)}}} + \underbrace{d_{N-2,2}}_{\substack{=f_{N-1} \\ \text{(by ind. hyp.)}}}$$

$$= f_N + f_{N-1} = f_{N+1}. \qquad \square$$

How do we compute f_n faster than by following recursion?

Thm. 1.5. (Binet's formula). For each $n \in \mathbb{N}$, we have

$$f_n = \frac{1}{\sqrt{5}} (\varphi^n - \psi^n),$$

where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1{,}618\ldots$

and $\psi = \frac{1-\sqrt{5}}{2} \approx -0.618\ldots$

What about $d_{n,m}$ for $m>2$?

Thm. 1.6. (Kasteleyn) Let m be even and $n\geq 1$.

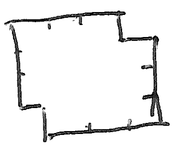
Then,
$$d_{n,m} = 2^{mn/2}\prod_{j=1}^{m/2}\prod_{k=1}^{n}\sqrt{\left(\cos\frac{j\pi}{m+1}\right)^2+\left(\cos\frac{k\pi}{n+1}\right)^2},$$

[Rmk: If $a_1, a_2, \dots, a_p$ are numbers, then $\prod_{i=1}^{p} a_i$ means $a_1\cdot a_2\cdot \dots \cdot a_p$.]

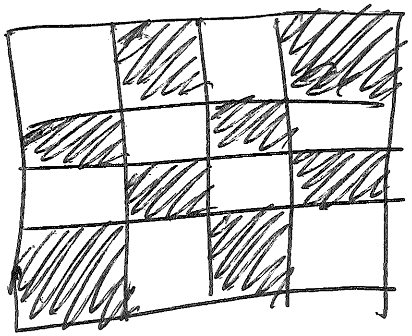
pf. see [Loehr, Thm. 12.85]. ← see the syllabus for the meaning of references such as [Loehr]

Ex: $d_{8,8} = 12\,988\,816$.

Exercise: For $n\geq 1$, let G_n be $R_{n,n}\setminus\{(1,1),(n,n)\}$.

(Ex: $G_4 =$  .) Prove that G_n has no domino tiling, when $n\geq 2$.

Sol: Color $R_{n,n}$ in a chessboard way:



i.e., each ~~for~~ (i,j) is black if $i+j$ is even,
white if $i+j$ is odd.

Any domino has exactly 1 black square and exactly 1 white square.

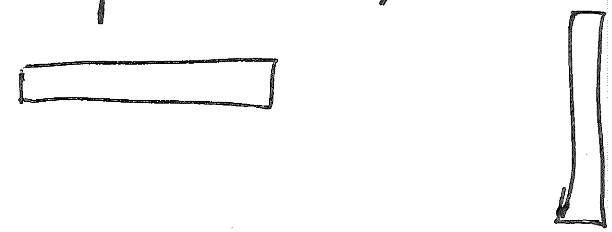
If we could tile G_n into dominos, then G_n would have equally many black and white squares.

But it does not, because:

- if n is even, # of whites = # of blacks + 2;
- if n is odd, the total # of squares is n^2-2, which is odd, so they cannot be evenly split into blacks & whites. $\lightning$

$\square$

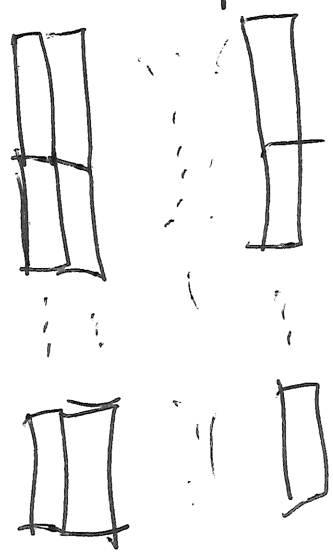
A <u>k-polymino</u> is a rectangle of shape $1 \times k$, or $k \times 1$.



<u>Prop. 1.7.</u> Let n, m, k be pos. integers. Then, $R_{n,m}$ can be tiled by k-polyminos if & only if $k | m$ or $k | n$.

<u>Proof.</u> $\Leftarrow$: Assume $k | m$ or $k | n$.

If $k | m$, then $R_{n,m}$ can be ~~covered by~~ tiled by vertical k-polyminos:



If $k | n$, then by horizontal ones.

$\Rightarrow$: Assume $R_{n,m}$ can be tiled by k-polyminos.
Consider k colors $0, 1, \dots, k-1$.
Color each square (i,j) with the color $(i+j) \% k$,
where $a \% b$ means the remainder of a when divided by b.

Each k-polymino has exactly 1 square of each color.
Since $R_{n,m}$ can be tiled, we thus conclude that each color appears equally often in $R_{n,m}$.
see [Ueck, p. 5–6] for why this is only possible when $k|n$ or $k|m$. $\square$