Def. A permutation σ of $[n] = \{1, 2, .., n\}$ is **short-legged** if $\forall\, i \in [n]$ we have $|\sigma(i) - i| \leq 1$.

Q: How many short-legged permutations are there?

Ex: For $n = 3$, these are

$(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $\cancel{(2,3,1)}$, $\cancel{(3,1,2)}$, $\cancel{(3,2,1)}$

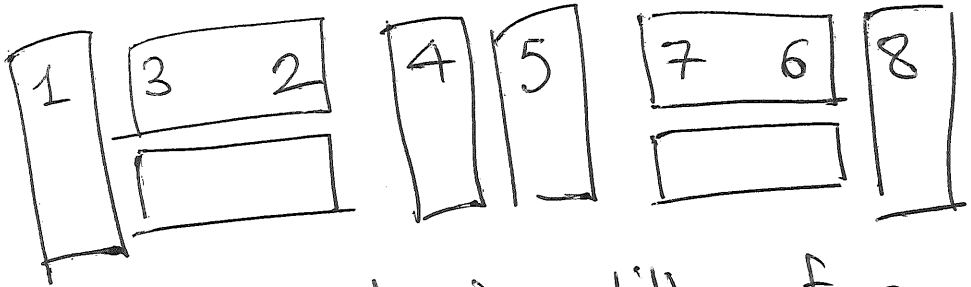
Ex: For $n = 4$, these are

$(1,2,3,4)$, $(1,2,4,3)$, $(1,3,2,4)$, $(2,1,3,4)$,
$(2,1,4,3)$.

Prop. 1.21. Let $n \in \mathbb{N}$. The # of short-legged permutations of $[n]$ is f_{n+1}.

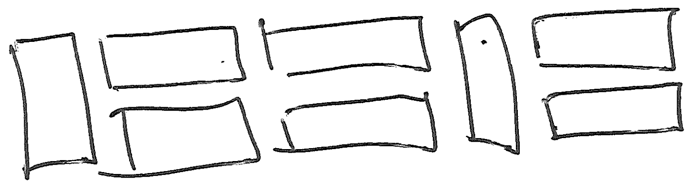
Proof idea. short-legged permutation of $[8]$:

$$\begin{array}{cccccccc} (1, & 3, & 2, & 4, & 5, & 7, & 6, & 8) \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$



This gives a domino tiling of a 2×8-rectangle.

Conversely:



$$\rightsquigarrow 1\ (3\ 2)\ (5\ 4)\ 6\ (8\ 7).$$

Thus, we have a bijection

$$\{\text{short-legged permutations of } [n]\} \to \{\text{domino tilings of } 2 \times n\text{-rectangle}\}.$$

But Prop. 1.4 shows that the # of the latter is f_{n+1}. $\square$

(Yes, this is informal.)

Counting subsets...

- An n-element set has 2^n subsets.
- An n-element set has $\binom{n}{k}$ k-elt subsets.

Def. A set S of integers is lacunar if it contains no two consecutive integers (i.e., $\nexists\, i \in \mathbb{Z}$ such that $i \in S$ and $i+1 \in S$).

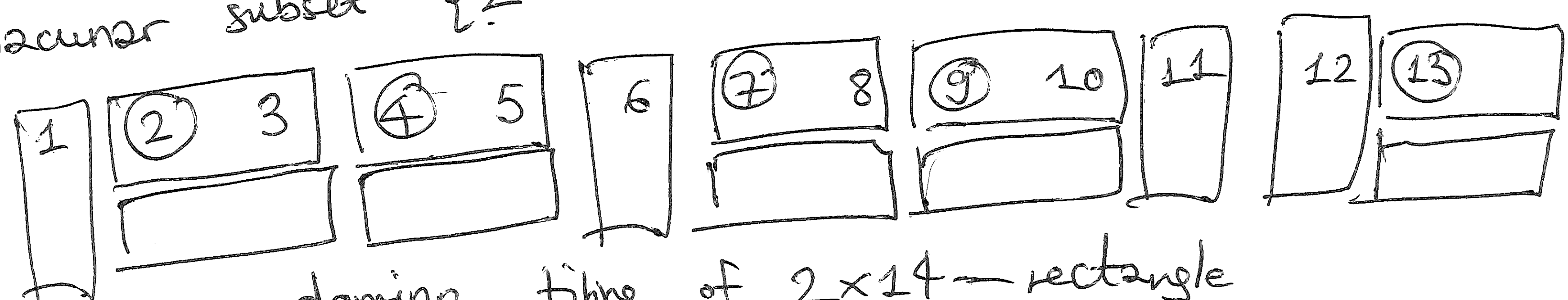
Q: How many lacunar subsets does $[n]$ have?

Prop. 1.22. Let $n \in \mathbb{N}$. The # of lacunar subsets of $[n]$ is f_{n+2}.

1st Proof idea. Induction.

2nd proof idea.

lacunar subset $\{2 < 4 < 7 < 9 < 13\}$ of $[13]$.

$\rightsquigarrow$ 

domino tiling of 2×14 — rectangle

Again, a bijection.

<u>Prop. 1.23.</u> Let $n \in \mathbb{N}$. The largest size of a lacunar subset of $[n]$ is $\lceil \frac{n}{2} \rceil$, where $\lceil x \rceil$ denotes the smallest integer $\geq x$ ("ceiling" of x, aka rounding up x).

<u>Proof.</u> The lacunar subset $\{1 < 3 < 5 < \ldots < (n \text{ or } n-1)\}$ has size $\lceil \frac{n}{2} \rceil$. So we only need to show that no higher size is possible.

Assume the contrary. So $\exists$ lacunar subset S of $[n]$ with $|S| > \lceil \frac{n}{2} \rceil$. Hence $|S| \geq \lceil \frac{n}{2} \rceil + 1 > \frac{n+1}{2}$.

Thus, $2|S| > n+1$.

Let S^+ be the set $\{s+1 \mid s \in S\}$.

(If $S = \{2, 4, 7, 10\}$, then $S^+ = \{3, 5, 8, 11\}$.)

Then, S and S^+ are two subsets of $[n+1]$. Thus,

$$|S \cup S^+| \leq n+1.$$

But S and S^+ are disjoint (since S is lacunar), so

$$|S \cup S^+| = |S| + |S^+| = |S| + |S| \quad (\text{since } |S^+| = |S|).$$

$$= 2|S| > n+1.$$

The previous 2 inequalities contradict each other. $\square$

2. Induction

We'll see many versions of induction. We start with the simplest one:

2.1. Standard induction $(0,\ n \to n+1)$.

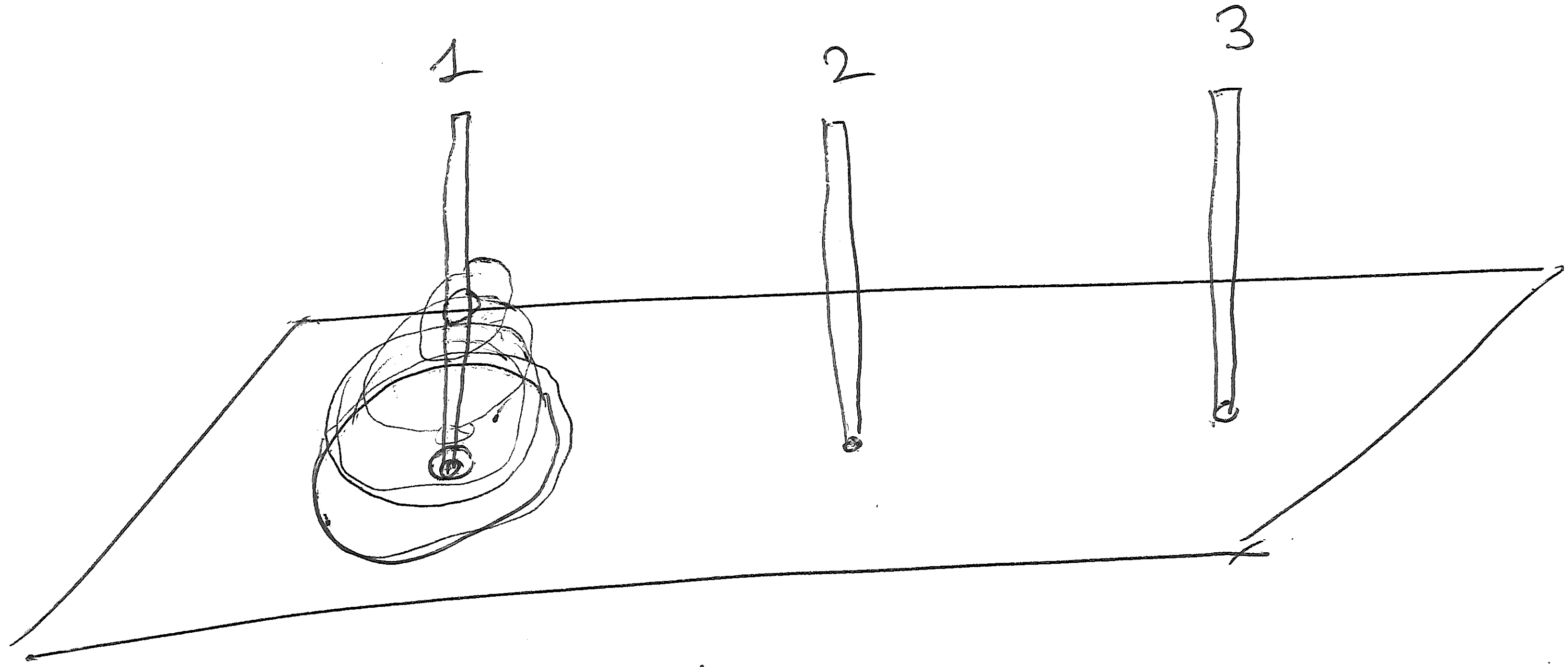
Induction principle 2.1. For each $n \in \mathbb{N}$, let $\mathcal{A}(n)$ be a statement.

Assume that

- $\mathcal{A}(0)$ is true,
- for each $n \in \mathbb{N}$: if $\mathcal{A}(n)$ is true, then $\mathcal{A}(n+1)$ is true.

Then, $\mathcal{A}(n)$ is true $\forall n \in \mathbb{N}$.

Example 2.2 (Towers of Hanoi): Given a wooden board, 3 pegs and n disks with little holes, so they can fit on the pegs.



The disks have different sizes,
originally all disks are wrapped around peg 1, with
the largest disk at the very bottom, the next-largest
a step further up, ...
In a single step, you can take the topmost
disk from a peg & move it to a different peg;

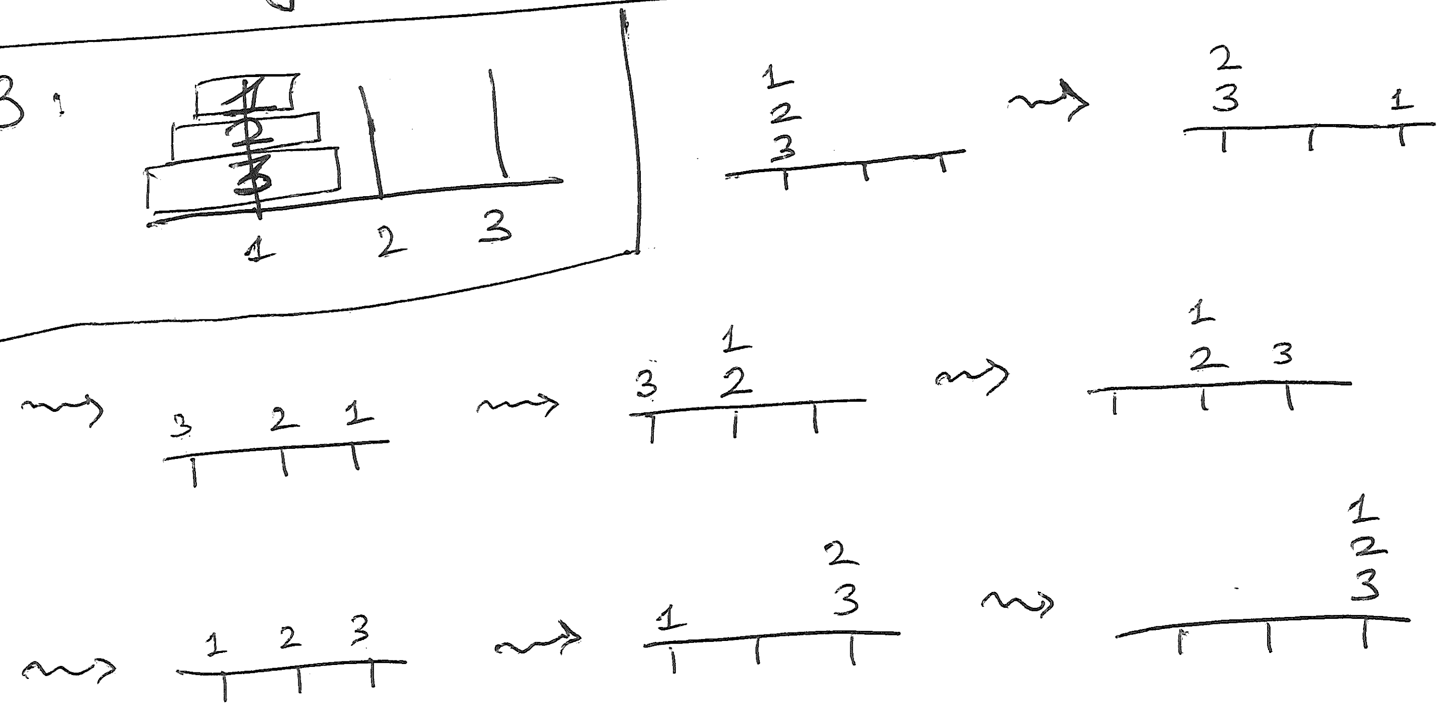
provided that it does not get placed on a smaller disk.

The goal is to move all disks to peg 3.

We call this "solving n-disk ToH".

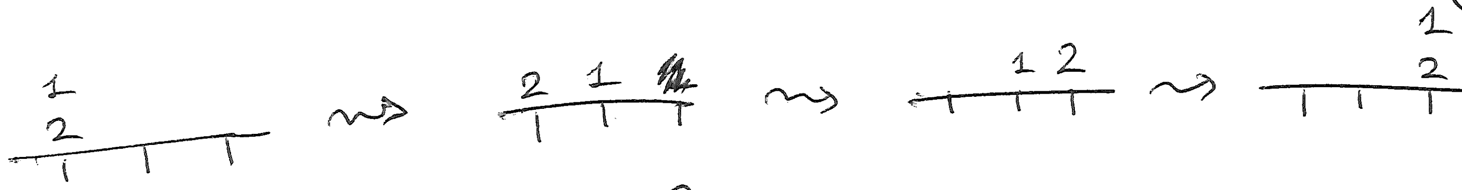
Q: Can we always solve n-disk ToH? In how many steps?

Ex: $n=3$:



Thus, the ~~x~~ 3-disk ToH ~~x~~ can be solved in 7 steps.

$n=2$:



The 2-disk ToH needs 3 steps.

The 1-disk ——— 1 step.

— 0-disk ——— 0 steps.

Prop. 2.3. Let $n \in \mathbb{N}$.
We can solve the n-disk ToH in $2^n - 1$ steps and cannot solve it in fewer steps.

Proof. Induction on n, i.e. use principle 2.1.

Let $\mathcal{A}(n) =$ "We can solve the n-disk ToH in $2^n - 1$ steps, and cannot solve it in fewer steps".

Then, $\mathcal{A}(0)$ is true.

Now, fix $n \in \mathbb{N}$. We want to prove that if $\mathcal{A}(n)$ is true, then $\mathcal{A}(n+1)$ is true.

Assume $\mathcal{A}(n)$ is true.

We must prove that $\mathcal{A}(n+1)$ is true, i.e. that the $(n+1)$-disk ToH can be solved in $2^{n+1}-1$ steps but not in fewer.

To see that it can be solved in $2^{n+1}-1$ steps, proceed as follows:

- First, move the disks $1, 2, \ldots, n$ to peg 2. This can be done in $2^n - 1$ steps (as it is just an n-disk ToH, so $\mathcal{A}(n)$ tells us it can be solved in $2^n - 1$ steps).
- Then, move disk $n+1$ to peg 3 (in 1 step).
- Then, move the disks $1, 2, \ldots, n$ to peg 3. This can be done in $2^n - 1$ steps (as it is again just an n-disk ToH).

Together, these are $(2^n - 1) + 1 + (2^n - 1) = 2^{n+1} - 1$ steps.

Why not in fewer?

Assume that you have some sequence ~~[illegible]~~ S of steps solving ~~the~~ $(n+1)$-disk ToH.

Somewhere in S, the disk $n+1$ must get moved.

Let P be the first time it moves, and Q the last time.

At step P, it moves from peg 1 to peg $k \in \{2,3\}$.

~~It was the only~~ Peg k is empty ~~at~~ that time, and peg 1 becomes empty after the move.

Thus, the remaining peg must contain the disks $1, 2, \ldots, n$.

Hence, ~~between the to~~ before step P, an n-disk ToH has been solved. Thus, by $\mathcal{A}(n)$, at least $2^n - 1$ steps were made.

A similar argument shows that at step Q, the disks $1, 2, \ldots, n$ all are on the same peg $\neq 3$.

Thus, after step Q, we still need to solve an n-disk ToH.

This, again, takes $\geq 2^n - 1$ steps.

Thus, we have made at least

$$\underbrace{(2^n-1)}_{\text{steps up to } P} + \underbrace{1}_{\text{step } P} + \underbrace{(2^n-1)}_{\text{steps after } Q} = 2^{n+1}-1$$

steps in S. So at least $2^{n+1}-1$ steps are needed.

Thus, $A(n+1)$ is proven.

So principle 2.1. says: $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

<u>Remark:</u> Moreover,

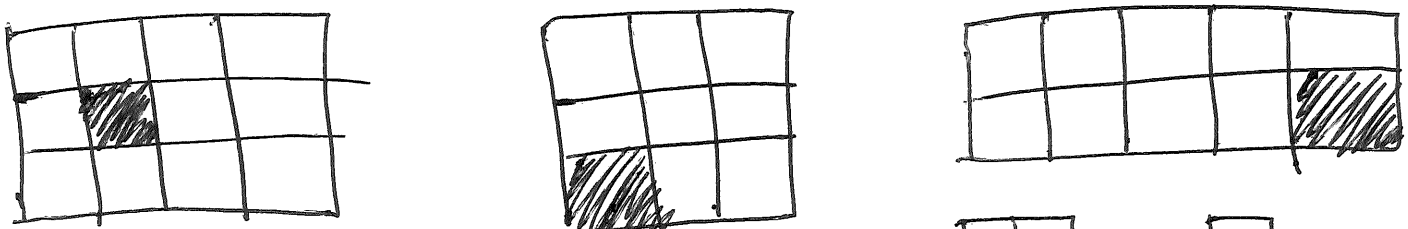
- there is exactly one sequence of 2^n-1 steps solving the n-disk ToH.

<u>Remark:</u> The shortest ~~sol~~ solution for ToH with >3 pegs is unknown.

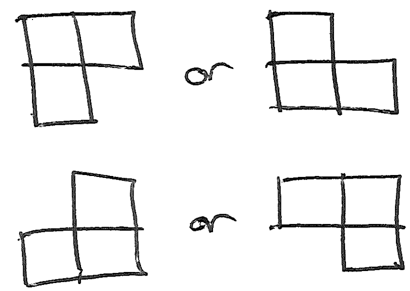
[LeLeMe, §16, 4.2].

Example 2.4. For positive integers p and q, a *mutilated $p\times q$-chessboard* is a $p\times q$-rectangle with one square missing.

Examples:



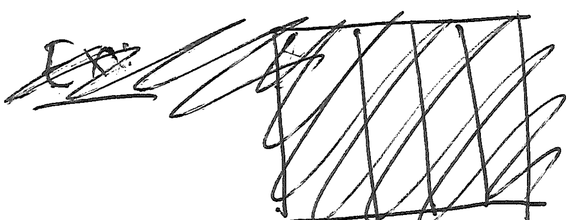
An *L-tile* is a "tile" of the form



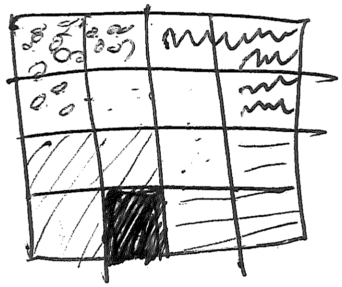
or ... or ... or ..., built of 3 1×1-rectangles.

Prop. 2.5 [LeLeMe, §5.1.5]. Let $n\in\mathbb{N}$. Any mutilated $2^n\times 2^n$-chessboard can be tiled with L-tiles.

~~Ex~~



Ex!



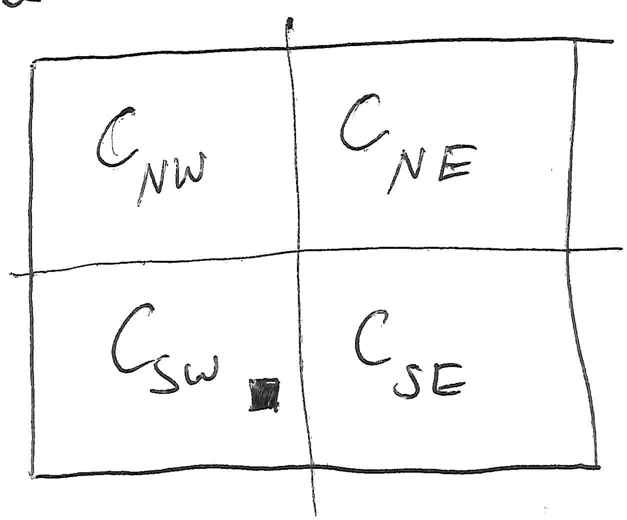
Proof. Use induction, i.e. apply Principle 2.1.

$A(n) =$ (any multilated $2^n \times 2^n$-chessboard can be tiled with L-tiles).

Then, $A(0)$ is true, since any multilated $2^0 \times 2^0$-chessboard is empty.

Now, fix $n \in \mathbb{N}$, and assume $A(n)$ is true.

Fix a multilated $2^{n+1} \times 2^{n+1}$-chessboard C.

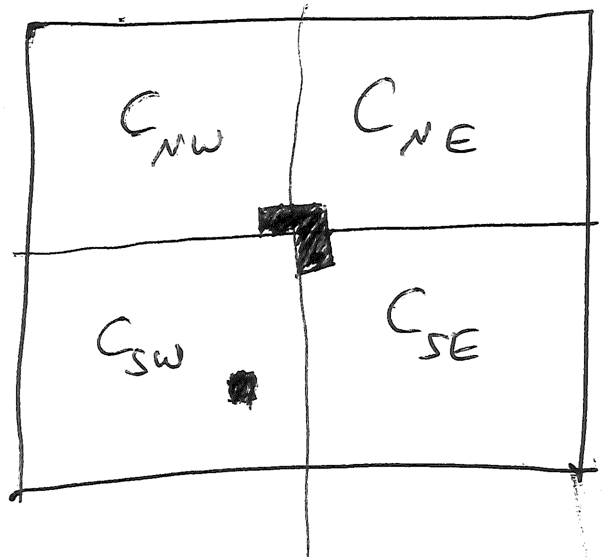


Split C along the vertical axes of symmetry.

Get 4 $2^n \times 2^n$-chessboards C_{NW}, C_{NE}, C_{SW}, C_{SE}, one of which is multilated.

Assume WLOG that C_{SW} is mutilated.

Thus, C_{SW} can be tiled with L-tiles (by $\mathcal{A}(n)$),



~~Remove three corner-squares, one for~~

Remove one corner-square from each of C_{NW}, C_{NE}, C_{SE} in such a way that these corners form an L-tile.

Now, C_{NW}, C_{NE}, C_{SE} become mutilated, and therefore can be tiled. Finally, combine these to a tiling of C.

So $\mathcal{A}(n+1)$ is true.

$\square$