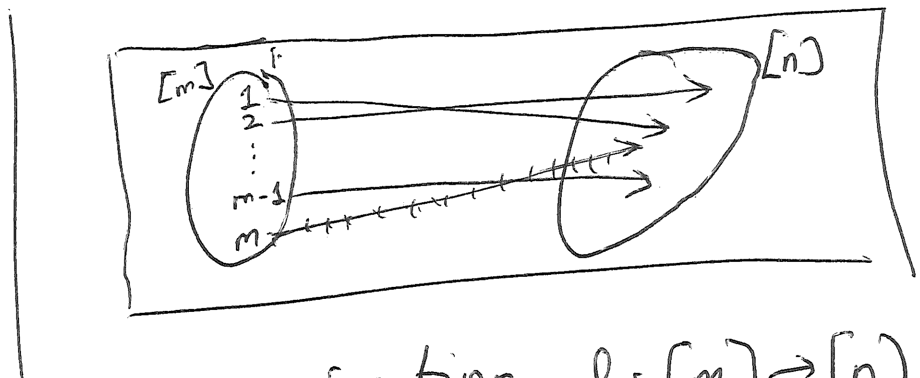
2nd approach. (continued). Recall: $m>0$ & $n>0$.

A surjection $f:[m]\to[n]$ is red if $f(m)=f(i)$ $\forall i<m$;
green otherwise.

Thus, if a surjection $f:[m]\to[n]$ is red, then $f|_{[m-1]}$ is "still" a surjection $[m-1]\to[n]$.



On the other hand, if a surjection $f:[m]\to[n]$ is green, then $f|_{[m-1]}$ has image $[n]\setminus\{f(m)\}$, so it can be viewed as a surjection $[m-1]\to[n]\setminus\{f(m)\}$.

This gives us an algorithm for constructing a red surjection $f:[m]\to[n]$:

- first, choose $f(m)$ (there are n choices);
- then, choose $f(1), f(2), \ldots, f(m-1)$; ~~these values~~ ~~should~~ in other words, choose $f|_{[m-1]}$ (there are $\mathrm{sur}(m-1, n)$ choices).

So there are $n \cdot \mathrm{sur}(m-1, n)$ <u>red</u> surjections $f : [m] \to [n]$.

We also get an algorithm for constructing a <u>green</u> surjection $f : [m] \to [n]$:

- first, choose $f(m)$ (there are n choices);
- then, choose $f(1), f(2), \ldots, f(m-1)$; in other words, choose $f|_{[m-1]}$ (there are $\mathrm{sur}(m-1, n-1)$ choices, because $f|_{[m-1]}$ needs to be a surjection $[m-1] \to [n] \setminus \{f(m)\}$).

So there are $n \cdot \mathrm{sur}(m-1, n-1)$ <u>green</u> surjections $f : [m] \to [n]$.

Now,

$\mathrm{sur}(m,n)$

$= (\#\text{ of surjections } [m] \to [n])$

$= \underbrace{(\#\text{ of red surjections } [m] \to [n])}_{= n \cdot \mathrm{sur}(m-1,n)} + \underbrace{(\#\text{ of green surjections } f : [m] \to [n])}_{= n \cdot \mathrm{sur}(m-1,n-1)}$

$= n \cdot (\mathrm{sur}(m-1,n) + \mathrm{sur}(m-1,n-1)).$

So we have proven:

__Prop. 3.12.__ Let m and n be positive integers. Then,

$$\mathrm{sur}(m,n) = n \cdot (\mathrm{sur}(m-1,n) + \mathrm{sur}(m-1,n-1)).$$

This also allows computing $\mathrm{sur}(m,n)$ recursively (using Prop. 3.10 (a) and (d)). We get the following analogue of Pascal's triangle:

n=0 n=1 n=2 n=3 n=4

m=0 →

1 0 0 0 0

0 1 0 0 0

0 1 2 0 0

0 1 6 6 0 0

0 1 14 36 24 0

0 1 30 150 240 120

Cor. 3.13. (a) $sur(n,n) = n!$ $\forall n \in \mathbb{N}$.

(b) $sur(m,n)$ is a multiple of $n!$ $\forall n \in \mathbb{N}$ & $m \in \mathbb{N}$.

Pf. (b) Easy by induction.

(a) Either by induction, or bijectively:
The surjections $[n] \to [n]$ are the permutations of $[n]$ (by the pigeonhole principle for surjections), so their number is $n!$. $\square$

Rmk. $sur(m,n)/n!$ is denoted $\left\{ {m \atop n} \right\}$, and is called a Stirling number of the 2nd kind.

Here is the most explicit formula for $sur(m,n)$ known:

Thm. 3.14. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Then,

$$sur(m,n) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i^m.$$

Proof. 1st proof: Use induction & one of the recursions above. (Specifically, Prop. 3.11.)
~~[illegible]~~ This is spelt out in [Fall 2017 Math 4707 HW 2 Exe 4].

2nd proof: Use the Principle of Inclusion & Exclusion. Later? $\square$

3.3. $1^m + 2^m + \cdots + n^m$

Next goal: prove Thm. 1.11. First step:

Thm. 3.15. Let $k \in \mathbb{N}$ and $m \in \mathbb{N}$. Then, $k^m = \sum_{i=0}^{m} \mathrm{sur}(m,i) \binom{k}{i}$.

Proof. How many ways are there to choose a map $f : [m] \to [k]$?

1st answer: k^m.

2nd answer: We choose f as follows:

- First, choose $|f([m])|$ (this is the size of the image of f, i.e., the # of distinct values of f).
  This is an integer in $\{0, 1, \ldots, m\}$. Call this integer i.
- Then, choose $f([m])$ (this is the SET of all values of f).
  There are $\binom{k}{i}$ choices for this (since $f([m])$ must be a i-element subset of $[k]$).

- Finally, choose $f(1), f(2), \ldots, f(m)$.
These must be chosen from the already determined i-element set $f([m])$, and must cover this set.
There are $\mathrm{sur}(m,i)$ choices here (since we're building a surjection from $[m]$ to the i-element set $f([m])$, which is already chosen).

So the total # of ways is $\sum_{i=0}^{m} \binom{k}{i} \mathrm{sur}(m,i)$.

Since the two answers answer the same question, we thus have $k^m = \sum_{i=0}^{m} \binom{k}{i} \mathrm{sur}(m,i)$. $\square$

(This was a proof by "double counting": answer a "how many" question in 2 ways, & compare the results.)

<u>Thm. 3.16.</u> Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Then,

$$\sum_{k=0}^{n} k^m = \sum_{i=0}^{m} \mathrm{sur}(m,i) \binom{n+1}{i+1}.$$

Proof.

$$\sum_{k=0}^{n} \underbrace{k^m}_{\substack{=\sum_{i=0}^{m} \mathrm{sur}(m,i)\binom{k}{i} \\ \text{(by Thm. 3.15)}}} = \underbrace{\sum_{k=0}^{n} \sum_{i=0}^{m} \mathrm{sur}(m,i)\binom{k}{i}}_{\substack{=\sum_{i=0}^{m}\sum_{k=0}^{n} \\ \text{(by Thm. 3.17} \\ \text{below)}}}$$

$$= \sum_{i=0}^{m} \sum_{k=0}^{n} \mathrm{sur}(m,i)\binom{k}{i} = \sum_{i=0}^{m} \mathrm{sur}(m,i) \underbrace{\sum_{k=0}^{n} \binom{k}{i}}_{\substack{=\binom{0}{i}+\binom{1}{i}+\dots+\binom{n}{i} \\ =\binom{n+1}{i+1} \text{ (by Thm. 1.16,} \\ \text{applied to } k=i)}}$$

$$= \sum_{i=0}^{m} \mathrm{sur}(m,i)\binom{n+1}{i+1}.$$

$\square$

We have interchanged two Σ signs in the above proof.
This relied on the following fact:

Thm. 3.17 (interchanging Σ signs / discrete Fubini theorem).

Let X and Y be finite sets. For any $x \in X$ and $y \in Y$, let $a_{x,y}$ be a number. Then,

$$\sum_{x \in X} \sum_{y \in Y} a_{x,y} = \sum_{(x,y) \in X \times Y} a_{x,y} = \sum_{y \in Y} \sum_{x \in X} a_{x,y}.$$

Example: If $X = [2]$ and $Y = [3]$, then this says

$$(a_{1,1} + a_{1,2} + a_{1,3}) + (a_{2,1} + a_{2,2} + a_{2,3})$$
$$= (\Sigma \text{ of all } a_{x,y})$$
$$= (a_{1,1} + a_{2,1}) + (a_{1,2} + a_{2,2}) + (a_{1,3} + a_{2,3}).$$

Idea: to sum the entries of a table, we can just as well sum the entries of each row & then sum the results, or do the same by column.

Remark. Thm. 3.17 is about <u>finite</u> sets X and Y.

If X and Y are infinite, then it may happen that

$$\sum_{x \in X} \sum_{y \in Y} a_{x,y} \neq \sum_{y \in Y} \sum_{x \in X} a_{x,y}$$

even if all these sums are well-defined.

Example: "serial debtor paradoxon": $X = Y = \mathbb{N}$

$a_{x,y}$	$y=0$	1	2	3	4	5
$x=0$	1	-1				
1		1	-1			
2			1	-1		
3				1	-1	
4					1	$\ddots$

(empty entries are 0's).

$$a_{x,y} = [x=y] - [x=y-1].$$

Then, $$\sum_{x\in X} \underbrace{\sum_{y \in Y} a_{x,y}}_{=0} = \sum_{x \in X} 0 = 0,$$

but $$\sum_{y\in Y} \underbrace{\sum_{x\in X} a_{x,y}}_{=[y=0]} = \sum_{y\in Y} [y=0] = 1.$$

For Thm. 3.17 to hold without finiteness of X and Y, we need to have $a_{x,y}=0$ for all but finitely many <u>pairs</u> $(x,y)\in X\times Y$. (There are also weaker conditions.)

<u>Proof of Thm. 1.11.</u> Rename k as m. Thus, we must prove
$$1^m + 2^m + \dots + n^m = \sum_{i=0}^{m} \operatorname{sur}(m,i) \cdot \binom{n+1}{i+1} \quad \text{for all } m>0,\ n\in\mathbb{N}.$$
But this is exactly what Thm. 3.16 yields, because
$$\sum_{k=0}^{n} k^m = 0^m + 1^m + 2^m + \dots + n^m = 1^m + 2^m + \dots + n^m \quad (\text{since } 0^m = 0). \quad \square$$

3.4. Vandermonde convolution

Thm. 3.18. Let $n \in \mathbb{N}$, $x \in \mathbb{Q}$, and $y \in \mathbb{Q}$. Then,

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \sum_{k} \binom{x}{k}\binom{y}{n-k}.$$

Rmk. The "$\sum_k$" in Thm. 3.18 means a sum over all $k \in \mathbb{Z}$.

Notice that $\binom{x}{k}\binom{y}{n-k} = 0$ whenever $k \notin \{0, 1, \ldots, n\}$

(indeed, $\binom{x}{k} = 0 \ \forall k < 0$, and $\binom{y}{n-k} = 0 \ \forall k > n$).

Thus, the 2nd equality in Thm. 3.18 is clear.

We'll first prove Thm. 3.18 under extra assumptions,

1st proof of Thm. 3.18 for the case when $x \in \mathbb{N}$.

$$\binom{u}{n} = \underbrace{\binom{u-1}{n-1}}_{=\binom{u-2}{n-2}+\binom{u-2}{n-1}} + \underbrace{\binom{u-1}{n}}_{=\binom{u-2}{n-1}+\binom{u-2}{n}}$$

$$= \underbrace{\binom{u-2}{n-2}}_{=\binom{u-3}{n-3}+\binom{u-3}{n-2}} + 2\underbrace{\binom{u-2}{n-1}}_{=\binom{u-3}{n-2}+\binom{u-3}{n-1}} + \underbrace{\binom{u-2}{n}}_{=\binom{u-3}{n-1}+\binom{u-3}{n}}$$

$$= \binom{u-3}{n-3} + 3\binom{u-3}{n-2} + 3\binom{u-3}{n-1} + \binom{u-3}{n}$$

$$= \binom{u-4}{n-4} + 4\binom{u-4}{n-3} + 6\binom{u-4}{n-2} + 4\binom{u-4}{n-1} + \binom{u-4}{n}$$

(notice that the coefficients are governed by the recurrence of Pascal's triangle, and hence are binomial coefficients)

$$= \cdots$$

$$= \binom{u-v}{n-v} + \cdots + \binom{v}{k}\binom{u-v}{n-k} + \cdots + \binom{u-v}{n}$$

$$= \sum_{k=0}^{v} \binom{v}{k}\binom{u-v}{n-k} \qquad \forall\, u \in \mathbb{Q} \text{ and } v \in \mathbb{N}.$$

Applying this to $u = x+y$ and $v = x$, we get

$$\binom{x+y}{n} = \sum_{k=0}^{x} \binom{x}{k}\binom{(x+y)-x}{n-k} = \sum_{k=0}^{x}\binom{x}{k}\binom{y}{n-k}$$

$$\overset{0}{=} \sum_{k\in\mathbb{Z}} \binom{x}{k}\binom{y}{n-k} \overset{0}{=} \sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}.$$

Here, the symbol "$\overset{0}{=}$" means that the left hand side & the right hand side differ only in addends that are 0, and thus are equal. Thus, Thm. 3.18 is proven for $x \in \mathbb{N}$. $\square$

2nd proof of Thm. 3.18 for the case when $x \in \mathbb{N}$ and $y \in \mathbb{N}$.

How many ways are there to choose an n-element subset of $\{1,2,\ldots,x\} \cup \{-1,-2,\ldots,-y\}$?

1st answer: $\binom{x+y}{n}$.

2nd answer: First, decide how many elements from $\{1,2,\ldots,x\}$ our set will have. Let this number be $k \in \{0,1,\ldots,n\}$.

Then, choose these k elements from $\{1,2,\ldots,x\}$ (this ~~allows~~ offers $\binom{x}{k}$ choices).

Then, choose the remaining $n-k$ elements for our subset from $\{-1, -2, \ldots, -y\}$ (this offers $\binom{y}{n-k}$ choices).

$\Longrightarrow$ The answer is $\sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k}$.

Now, compare the two answers. ("Double counting" again!)
This proves Thm. 3.18 when $x \in \mathbb{N}$ and $y \in \mathbb{N}$. $\square$

3rd proof for all x, y. See [detnotes, second proof of Thm. 2.27].
(It proceeds by induction on n.)

Remark. In Thm. 3.18, we can replace "$\sum_{k=0}^{n}$" by "$\sum_{k=0}^{x}$" when $x \in \mathbb{N}$. (We've already seen this in the 1st proof.)

It turns out that just one extra trick suffices to make the first two proofs above yield Thm. 3.18 for ALL $x, y \in \mathbb{Q}$.
This is the "polynomial identity trick". We'll see this next time.

<u>Cor. 3.19.</u> Let $n \in \mathbb{N}$. Then,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

<u>Proof of Cor. 3.19.</u> Thm. 3.18 (applied to $x=n$ & $y=n$) yields

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k} \underbrace{\binom{n}{n-k}}_{\substack{=\binom{n}{k} \\ \text{(by symmetry)}}} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2. \qquad \square$$

<u>Rmk.</u> There is no known formula for $\sum_{k=0}^{n} \binom{n}{k}^3$.

However, there are such formulas for

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3.$$

We'll see the first of these soon.