3.2. COUNTING MAPS

How many maps are there from a set to another?

Thm. 3.4. Let $m, n \in \mathbb{N}$. Let A be an m-element set. Let B be an n-element set. Then,

$$(\# \text{ of maps from } A \text{ to } B) = n^m.$$

Proof. Informally: Choosing a map f from A to B means (independently) choosing an image for each element of A. Thus, there are n^m ways to do this (since A has m elements, and for each we have n choices).

More rigorously: Let $(a_1, a_2, \ldots, a_m)$ be a list of all elements of A (with no repetitions). Then,

$$\begin{aligned} \{\text{maps } A \to B\} &\longrightarrow B^m, \\ f &\longmapsto (f(a_1), f(a_2), \ldots, f(a_m)) \end{aligned}$$

is a bijection. Thus, $|\{\text{maps } A \to B\}| = |B^m| = |B|^m = n^m$, since $|B| = n$. $\square$

Thm. 3.5. Let $m, n \in \mathbb{N}$. Let A be an m-element set. Let B be an n-element set. Then,

$$(\#\text{ of injective maps from } A \text{ to } B) = n(n-1)\cdots(n-m+1).$$

Remark. (a) If $m=0$, then the RHS is the empty product, thus $=1$. And indeed, there is exactly one injective map from A to B (namely, the "empty map": it doesn't have to send anything anywhere, since A has no elements).

(b) If $m>n$, then the LHS is 0 by the pigeonhole principle for injections (Thm. 3.6(a) below), while the RHS is 0 since the product contains $n-n=0$ as a factor.

1st proof of Thm. 3.5. (informal). ~~Choosing an injective map f from~~ Let $(a_1, a_2, \ldots, a_m)$ be a list of all elements of A (with no repetitions).

Then, ~~choosing~~ an injective map $f: A \to B$ can be constructed as follows:

- choose $f(a_1)$ (there are n options ~~here~~ for this, since $f(a_1)$ should lie in B);
- choose $f(a_2)$ to be distinct from $f(a_1)$ (there are $n-1$ options for this);
- choose $f(a_3)$ to be distinct from $f(a_1)$ and $f(a_2)$ (there are $n-2$ options for this, since $f(a_1)$ and $f(a_2)$ are distinct);
- etc. (last step: choose $f(a_m)$ (there are $n-m+1$ options)).

In total, ~~by the~~ we thus have $n(n-1)(n-2)\cdots(n-m+1)$ options. $\square$

I'm going to show two more proofs, which essentially just make the 1st proof more rigorous.

I'll use the notation $\mathrm{Inj}(A,B)$ for the set of all injective maps from A to B.

So Thm. 3.5 states that $|\mathrm{Inj}(A,B)| = n(n-1)\cdots(n-m+1)$

(where A, B, m, n are as in Thm. 3.5).

2nd proof of Thm. 3.5. (explicit bijection).

Let $(a_1, a_2, \ldots, a_m)$ be a list of all elements of A.
Furthermore, WLOG assume that $B = [n]$
(since $|\mathrm{Inj}(A,B)| = |\mathrm{Inj}(A,[n])|$, because we can relabel the elements of B as $1, 2, \ldots, n$
— like in the proof of Lemma 1.22).

Next, set $G = [n] \times [n-1] \times \cdots \times [n-m+1]$, where
$[k] = \{1, 2, \ldots, k\}$ for $k \in \mathbb{Z}$ (so $[k] = \emptyset$ if $k \leq 0$),

Observation 1: $|G| = n(n-1) \cdots (n-m+1)$.

[Proof: If $m > n$, then both sides are 0 (since one of the factors in G is $\emptyset$).
Else, $|G| = |[n] \times [n-1] \times \cdots \times [n-m+1]|$

$$= \underbrace{|[n]|}_{=n} \cdot \underbrace{|[n-1]|}_{=n-1} \cdot \ldots \cdot \underbrace{|[n-m+1]|}_{=n-m+1}$$

$$= n(n-1) \cdots (n-m+1). \qquad \square \;]$$

Now, we shall construct maps $\mathrm{Inj}(A,B) \to G$ and $G \to \mathrm{Inj}(A,B)$ that are mutually inverse.

- To define a map $L : \mathrm{Inj}(A,B) \to G$, we send $f \in \mathrm{Inj}(A,B)$ to $(g_1, g_2, \ldots, g_m) \in G$,

  where g_i is such that $f(a_i)$ is the g_i-th smallest element of $B \setminus \{f(a_1), \ldots, f(a_{i-1})\}$.

  (This L is called the <u>Lehmer encoding</u>: for each $f \in \mathrm{Inj}(A,B)$, the tuple $L(f) = (g_1, g_2, \ldots, g_m)$ is called the <u>Lehmer code</u> of f.)

- To define a map $M : G \to \mathrm{Inj}(A,B)$, we send $(g_1, g_2, \ldots, g_m) \in G$ to the injective map $f \in \mathrm{Inj}(A,B)$ constructed recursively as follows:

  If $f(a_1), f(a_2), \ldots, f(a_{i-1})$ are already defined, then $f(a_i)$ is defined to be the g_i-th smallest element of $B \setminus \{f(a_1), f(a_2), \ldots, f(a_{i-1})\}$.

It is easy to see (formally: strong induction) that L and M are well-defined & mutually inverse.
So L is a bijection. Thus,

$$|\operatorname{Inj}(A,B)| = |G| = n(n-1)\cdots(n-m+1). \qquad \square$$

3rd proof (induction). Induction on m.

Base: $m=0$, so that $A=\emptyset$. Trivial (see Remark above).

Step: Let $k\in\mathbb{N}$. Assume (as ind. hyp.) that Thm. 3.5 holds for $m=k$. Now let's prove it for $m=k+1$.
~~Fix $a\in A$ (this exists since~~ Let ~~A, B, m~~ A be a $(k+1)$-element set, and B an n-element set for some $n\in\mathbb{N}$.
We must show $|\operatorname{Inj}(A,B)| = n(n-1)\cdots(n-(k+1)+1)$.
Fix $a\in A$ (this exists, since $|A|=k+1\geq 1>0$.).
The set $A\setminus\{a\}$ has $(k+1)-1=k$ elements.
Thus, by the ~~[illegible]~~ ind. hyp.,

(1) $\quad |\mathrm{Inj}(A \setminus \{a\}, B)| = n(n-1)\cdots(n-k+1).$

Now, consider the map

$$R : \mathrm{Inj}(A, B) \to \mathrm{Inj}(A \setminus \{a\}, B),$$
$$f \mapsto f|_{A \setminus \{a\}}$$

(well-defined for obvious reasons).

Observation 1: Let $g \in \mathrm{Inj}(A \setminus \{a\}, B)$. Then, there are precisely $n-k$ maps $f \in \mathrm{Inj}(A, B)$ such that $R(f) = g$.

[Proof of Observation 1: To construct a map $f \in \mathrm{Inj}(A, B)$ such that $R(f) = g$, we only need to choose a value for $f(a)$. This value must be distinct from all values of g (in order for f to be injective); these are precisely k values to avoid (because g is injective, thus takes k distinct values). Thus, in total, there are $n-k$ options for $f(a)$. This proves Obs. 1.]

Now,

$$
\begin{aligned}
&|\operatorname{Inj}(A,B)| \\
&= (\text{\# of all } f \in \operatorname{Inj}(A,B)) \\
&= \sum_{g \in \operatorname{Inj}(A \setminus \{a\}, B)} \underbrace{(\text{\# of all } f \in \operatorname{Inj}(A,B) \text{ such that } R(f) = g)}_{= n-k} \\
&= \sum_{g \in \operatorname{Inj}(A \setminus \{a\}, B)} (n-k) = \underbrace{|\operatorname{Inj}(A \setminus \{a\}, B)|}_{\substack{= n(n-1)\cdots(n-k+1) \\ (\text{by } (1))}} \cdot (n-k) \\
&= n(n-1)\cdots(n-k+1) \cdot (n-k) \\
&= n(n-1)\cdots(n-k) = n(n-1)\cdots(n-(k+1)+1).
\end{aligned}
$$

This concludes the induction step. Thus, Thm. 3.5 is proven. □

Thm. 3.6 (Pigeonhole principle for injections).

Let $f: A \to B$ be an injective map between finite sets.

Then: (a) $|A| \leqslant |B|$.

(b) If $|A| = |B|$, then f is bijective.

Remark: This is false if A, B are not finite. (More precisely, (b) is false). For example, the map

$$\mathbb{N} \to \mathbb{N}, \quad i \mapsto i+1$$

is injective, but NOT bijective (0 is not in its image).

Thm. 3.7. (Pigeonhole principle for surjections).

Let $f: A \to B$ be a surjective map between finite sets.

Then: (a) $|A| \geqslant |B|$.

(b) If $|A| = |B|$, then f is bijective.

Remark: Again, (b) is false for infinite sets.

For example, the map $\mathbb{N} \to \mathbb{N}$, $\quad i \mapsto \begin{cases} i-1 & \text{if } i > 0; \\ 0 & \text{if } i = 0 \end{cases}$

is surjective, but NOT bijective.

Cor. 3.8. Let X be a finite set.

Then, (# of permutations of X) $= |X|!$.

Proof. Thm. 3.6 (b) shows that every injective map from X to X is bijective. The converse also holds. Thus,

{injective maps from X to X}
$= \{$bijective maps from X to $X\} = \{$permutations of $X\}$.

Hence,
(# of injective maps from X to X) = (# of permutations of X).
But the LHS is $|X|(|X|-1)\cdots\underbrace{(|X|-|X|+1)}_{=1}$

$$= |X|(|X|-1)\cdots 1 = |X|!$$

(by Thm. 3.5, applied to $m=|X|$, $n=|X|$, $A=X$ and $B=X$).
Thus, (# of permutations of X) $= |X|!$. □

Next, we shall count <u>surjective</u> maps $A \to B$.

<u>Def.</u> Let $m, n \in \mathbb{N}$. Then, $\operatorname{sur}(m,n)$ means the # of surjective maps from $[m]$ to $[n]$.

<u>Prop. 3.9.</u> Let $m, n \in \mathbb{N}$. Let A be an m-element set. Let B be an n-element set. Then,

(# of surjective maps $A \to B$) $= \operatorname{sur}(m,n)$.

<u>Proof.</u> Relabel the elements of A as $1, 2, \ldots, m$, and relabel — // — B as $1, 2, \ldots, n$.

(Like in the proof of lemma 1.22) $\square$

What can we say about $\operatorname{sur}(m,n)$?

<u>Prop. 3.10.</u> (a) $\operatorname{sur}(m,0) = [m=0]$. $\forall m \in \mathbb{N}$.

(b) $\operatorname{sur}(m,1) = [m \neq 0] = 1 - [m=0]$. $\forall m \in \mathbb{N}$.

(c) $\text{sur}(m,2) = 2^m - 2 + [m=0]$. $\forall m \in \mathbb{N}$.

(d) $\text{sur}(0,k) = [k=0]$ $\forall k \in \mathbb{N}$,

(e) $\text{sur}(1,k) = [k=1]$ $\forall k \in \mathbb{N}$,

~~Example: sur(2,3)~~

(f) $\text{sur}(m,n) = 0$ $\forall m < n$.

Proof.
(a) If $m \neq 0$, then $\nexists$ surjections $[m] \to [0]$, since the arrows have nowhere to point.
If $m=0$, then there is one.

(b) There is always exactly 1 map $[m] \to [1]$.
It is surjective if & only if $m \neq 0$.

(c) There are 2^m maps $[m] \to [2]$.
Exactly 2 of them are non-surjective, unless $m=0$.

(d), (e) LTTR ($\leftarrow$ left to the reader).

(f) Thm. 3.7(a) shows that no surjections $[m] \to [n]$ exist. $\square$

Let's look out for a recursive formula for $\operatorname{sur}(m,n)$.

1st approach: Fix $m \in \mathbb{N}$ and $n > 0$.

Given a surjective map $f : [m] \to [n]$, we let J_f be the set of all $i \in [m]$ such that $f(m) = n$.

Clearly, $J_f \neq \emptyset$. Thus,

$$(\# \text{ of all surjective maps } f : [m] \to [n])$$

$$= \sum_{\substack{J \subseteq [m];\\ J \neq \emptyset}} \underbrace{(\# \text{ of all surjective maps } f : [m] \to [n] \text{ with } J_f = J)}_{= (\# \text{ of all surjective maps } [m] \setminus J \to [n-1])}$$

(because there is a bijection $\{\text{surjective maps } f : [m] \to [n] \text{ with } J_f = J\} \to \{\text{surjective maps } [m] \setminus J \to [n-1]\}$, which sends each f to $f|_{[m] \setminus J}$)

$$= \sum_{\substack{J \subseteq [m];\\ J \neq \emptyset}} \underbrace{(\# \text{ of all surjective maps } [m] \setminus J \to [n-1])}_{= \operatorname{sur}(|[m] \setminus J|,\ n-1) \text{ (by Prop. 3.9)}}$$

$$= \sum_{\substack{J \subseteq [m]; \\ J \neq \varnothing}} \operatorname{sur}\big(\underbrace{|[m] \setminus J|}_{= m - |J|},\ n-1\big)$$

$$= \sum_{\substack{J \subseteq [m]; \\ J \neq \varnothing}} \operatorname{sur}(m - |J|,\ n-1)$$

$$= \sum_{j=1}^{m} \operatorname{sur}(m-j,\ n-1) \cdot \underbrace{(\#\text{ of } J \subseteq [m] \text{ with } J \neq \varnothing \text{ satisfying } |J| = j)}_{\substack{= (\#\text{ of } J \subseteq [m] \text{ satisfying } |J| = j) \\ (\text{since } j > 0) \\ = \binom{m}{j}}}$$

$$= \sum_{j=1}^{m} \operatorname{sur}(m-j,\ n-1) \cdot \binom{m}{j} = \sum_{j=1}^{m} \binom{m}{j} \operatorname{sur}(m-j,\ n-1).$$

Since the LHS is $\text{sur}(m,n)$, we thus have proven the following:

<u>Prop. 3.11.</u> Let $m \in \mathbb{N}$ and $n > 0$. Then,

$$\text{sur}(m,n) = \sum_{j=1}^{m} \binom{m}{j} \text{sur}(m-j, n-1)$$

$$= \sum_{j=0}^{m-1} \underbrace{\binom{m}{m-j}}_{=\binom{m}{j}} \text{sur}(\underbrace{m-(m-j)}_{=j}, n-1)$$

(we substituted $m-j$ for j here)

$$= \sum_{j=0}^{m-1} \binom{m}{j} \text{sur}(j, n-1).$$

Prop. 3.11 + Prop. 3.10 (d) allow you to compute $\text{sur}(m,n)$'s one by one, recursively.

<u>2nd approach:</u> Fix $m > 0$ and $n > 0$.

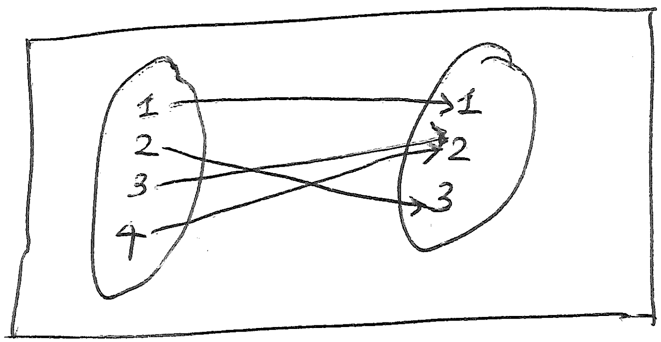
Classify the surjections according to the image of m.

A surjection $f : [m] \to [n]$ is called

- <u>red</u> if $f(m) = f(i)$ for some $i < m$;
- <u>green</u> if it is not red (i.e., if $f(m) \neq f(i)\ \forall i < m$).

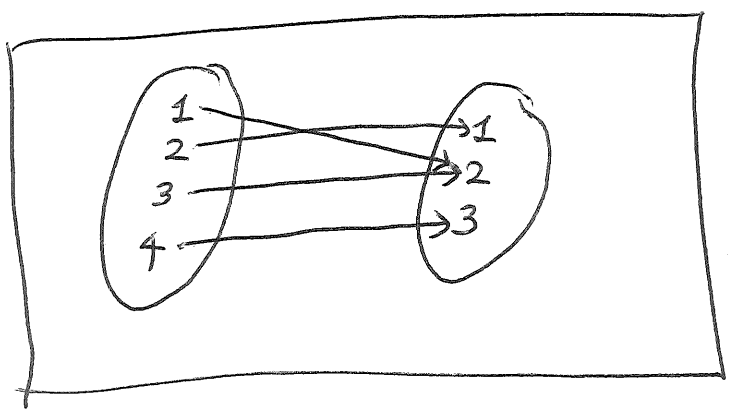
~~[to be continued!]~~

[<u>Examples:</u> If $m = 4$ and $n = 3$, then



is red,

while



is green.]

[to be continued!]