## 2.5. Strong induction with "bases"

<u>Induction principle 2.22.</u> Let $g \in \mathbb{Z}$ and $k \in \mathbb{Z}$. ~~...~~
Let $\mathcal{A}(n)$ be a logical statement for all $n \in \mathbb{Z}_{\geq g}$.
Assume that:

- $\mathcal{A}(g), \mathcal{A}(g+1), \ldots, \mathcal{A}(k-1)$ hold.
- $\forall n \in \mathbb{Z}_{\geq k}$, if ($\mathcal{A}(m)$ holds $\forall m < n$), then $\mathcal{A}(n)$ holds.

Then, $\mathcal{A}(n)$ holds $\forall n \in \mathbb{Z}_{\geq g}$.

<u>Proof.</u> Follows easily from Induction principle 2.16. $\square$

## 2.6. Two-sided induction

<u>Induction principle 2.23.</u> For all $n \in \mathbb{Z}$, let $\mathcal{A}(n)$ be a logical statement.

Assume that:

- ~~...~~ $\mathcal{A}(0)$ holds.
- $\forall n \in \mathbb{N}$: if $\mathcal{A}(n)$ holds, then $\mathcal{A}(n+1)$ holds.
- $\forall n \in -\mathbb{N}$: if $\mathcal{A}(n)$ holds, then $\mathcal{A}(n-1)$ holds.

Here, $-\mathbb{N} := \{0, -1, -2, \ldots\}$.

Then, $\mathcal{A}(n)$ holds $\forall n \in \mathbb{Z}$.

<u>Example:</u> Thm. 1.14 (stating that $\binom{n}{k} \in \mathbb{Z}$, whenever $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$) can be proven by Induction principle 2.23.

## 3. Binomial coefficients

In this chapter: closer look at binomial coefficients

References:

- [detnotes, ch. 2] $\leftarrow$ basics in detail
- [GKP, ch. 5] $\leftarrow$ best treatment of bin. coeffs.

### 3.1. Basic identities.

See §1.3.

- Definition: $\binom{n}{k} = \dfrac{n(n-1)\cdots(n-k+1)}{k!}$ for $k \in \mathbb{N}$.

  $\binom{n}{k} = 0$ for $k \notin \mathbb{N}$.

  ($n$ can be any number.)

- $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ holds only when $n \in \mathbb{N}$, $k \in \mathbb{N}$ and $n \geq k$.

- Prop. 1.11a, ..., Prop. 1.20.

Recall Thm. 1.16: Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then,

$$\binom{0}{k} + \binom{1}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}.$$

<u>Remark.</u> The first $k$ addends on the LHS are $0$.

So we get $\binom{k}{k} + \binom{k+1}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}$.

<u>1st proof of Thm. 1.16:</u>

Example:

$$\binom{8}{3} = \binom{7}{2} + \underbrace{\binom{7}{3}}_{=\binom{6}{2}+\binom{6}{3}} = \binom{7}{2} + \binom{6}{2} + \underbrace{\binom{6}{3}}_{=\binom{5}{2}+\binom{5}{3}}$$

$$= \binom{7}{2} + \binom{6}{2} + \binom{5}{2} + \underbrace{\binom{5}{3}}_{=\binom{4}{2}+\binom{4}{3}}$$

$$= \binom{7}{2} + \binom{6}{2} + \binom{5}{2} + \binom{4}{2} + \underbrace{\binom{4}{3}}_{=\binom{3}{2}+\binom{3}{3}}$$

$$= \binom{7}{2} + \binom{6}{2} + \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \underbrace{\binom{3}{3}}_{=\binom{2}{2}+\cancel{\binom{2}{3}}}$$

$$= \binom{7}{2} + \binom{6}{2} + \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}.$$

In general:

$$\binom{n+1}{k+1} = \binom{n}{k} + \underbrace{\binom{n}{k+1}}_{=\binom{n-1}{k}+\binom{n-1}{k+1}} = \binom{n}{k} + \binom{n-1}{k} + \underbrace{\binom{n-1}{k+1}}_{=\binom{n-2}{k}+\binom{n-2}{k+1}}$$

$$= \cdots = \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k}{k} + \underbrace{\binom{k}{k+1}}_{=0}$$

$$= \binom{n}{k} + \binom{n-1}{k} + \cdots + \binom{k}{k}.$$

$\square$

To make this rigorous: induction on $n$.

<u>2nd proof of Thm. 1.16:</u>

For each $i \in \mathbb{N}$, we have $\binom{i}{k} = \binom{i+1}{k+1} - \binom{i}{k+1}$

(since $\binom{i+1}{k+1} = \binom{i}{k} + \binom{i}{k+1}$).

Hence,

$$\binom{0}{k}+\binom{1}{k}+\dots+\binom{n}{k}$$

$$=\left(\cancel{\binom{1}{k+1}}-\binom{0}{k+1}\right)+\left(\cancel{\binom{2}{k+1}}-\cancel{\binom{1}{k+1}}\right)+\left(\cancel{\binom{3}{k+1}}-\cancel{\binom{2}{k+1}}\right)$$

$$\cancel{+\dots+}\left(\cancel{\binom{n}{k+1}}-\cancel{\binom{n-1}{k+1}}\right)+\left(\binom{n+1}{k+1}-\cancel{\binom{n}{k+1}}\right)$$

$$=\binom{n+1}{k+1}-\underbrace{\binom{0}{k+1}}_{=0}=\binom{n+1}{k+1}. \qquad \square$$

The cancellations in the 2nd proof are an instance of the telescope principle:

<u>**Prop. 3.1.** (Telescoping sum principle).</u> Let $u$ and $v$ be integers with $u \leq v+1$. ~~Then~~ Let $a_u, a_{u+1}, \dots, a_{v+1}$ be numbers. Then,

$$\sum_{j=u}^{v}\left(a_{j+1}-a_j\right)=a_{v+1}-a_u.$$

This is the discrete version of the fundamental theorem of calculus ( $\int_u^v f'(t)\,dt = f(v) - f(u)$ ).

<u>3rd proof of Thm. 1.16.</u> Thm. 1.15 yields

$$\binom{n+1}{k+1} = (\#\text{ of } (k+1)\text{-elt. subsets of } [n+1])$$

$$(1) \qquad = \sum_{j=1}^{n+1} (\#\text{ of } (k+1)\text{-elt. subsets of } [n+1] \text{ whose largest element is } j).$$

Now fix $j \in [n+1]$. How many $(k+1)$-elt. subsets of $[n+1]$ are there whose largest elt. is $j$? The answer is $\binom{j-1}{k}$, since the element $j$ has already been chosen, and the remaining $k$ elts. must be chosen from $\{1, 2, \ldots, j-1\} = [j-1]$, which can be done in $\binom{j-1}{k}$ ways.

[ Rigorous version:

There is a bijection

$$\{k\text{-elt. subsets of } [j-1]\} \to \{(k+1)\text{-elt. subsets of } [n+1) \text{ whose largest elt. is } j\},$$

$$S \mapsto S \cup \{j\}$$

(the inverse map sends $T$ to $T \setminus \{j\}$).

(We have to prove that these two maps are well-defined & mutually inverse. But this is easy.)

Thus, $|\{(k+1)\text{-elt. subsets of } [n+1] \text{ whose largest elt. is } j\}|$
$= |\{k\text{-elt. subsets of } [j-1]\}|$
$= (\#\text{ of } k\text{-elt. subsets of } [j-1]) = \binom{j-1}{k}$ (by Thm. 1.15).

So this proves our answer. ]

So (1) becomes

$$\binom{n+1}{k+1} = \sum_{j=1}^{n+1} \binom{j-1}{k} = \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n}{k}. \qquad \square$$

<u>Prop. 3.2. (Trinomial revision).</u> Let $n, a, b \in \mathbb{R}$. Then,

$$\binom{n}{a}\binom{a}{b} = \binom{n}{b}\binom{n-b}{a-b}.$$

<u>Proof of Prop. 3.2.</u> Four cases:

Case 1: $b \notin \mathbb{N}$,

Case 2: $b \in \mathbb{N}$ and $a \notin \mathbb{N}$,

Case 3: $b \in \mathbb{N}$ and $a \in \mathbb{N}$ but $a < b$.

Case 4: $b \in \mathbb{N}$ and $a \in \mathbb{N}$ and $a \geq b$.

Most are rather obvious:

- In Case 1, we must prove $\binom{n}{a} 0 = 0 \binom{n-b}{a-b}$.
- In Case 2, we must prove $0 \binom{a}{b} = \binom{n}{b} 0$

  (indeed, from $b \in \mathbb{N}$ and $a \notin \mathbb{N}$, we get $a - b \notin \mathbb{N}$, so $\binom{n-b}{a-b} = 0$).

- In Case 3, we must prove $\binom{n}{a} 0 = \binom{n}{b} 0$

  (indeed, the conditions of Case 3 force $\binom{a}{b} = 0$, but also $a < b \Rightarrow a - b \notin \mathbb{N} \Rightarrow \binom{n-b}{a-b} = 0$).

- In Case 4, we have $a-b \in \mathbb{N}$. Now,

$$\binom{n}{b}\binom{n-b}{a-b} = \frac{n(n-1)\cdots(n-b+1)}{b!} \cdot \frac{(n-b)(n-b-1)\cdots((n-b)-(a-b)+1)}{(a-b)!}$$

$$= \frac{n(n-1)\cdots\cdots\cdots((n-\cancel{b})-(a-\cancel{b})+1)}{b!\,(a-b)!}$$

$$= \frac{n(n-1)\cdots(n-a+1)}{b!\,(a-b)!}.$$

Compared with

$$\binom{n}{a}\binom{a}{b} = \frac{n(n-1)\cdots(n-a+1)}{\cancel{a!}} \cdot \frac{\cancel{a!}}{b!\,(a-b)!}$$

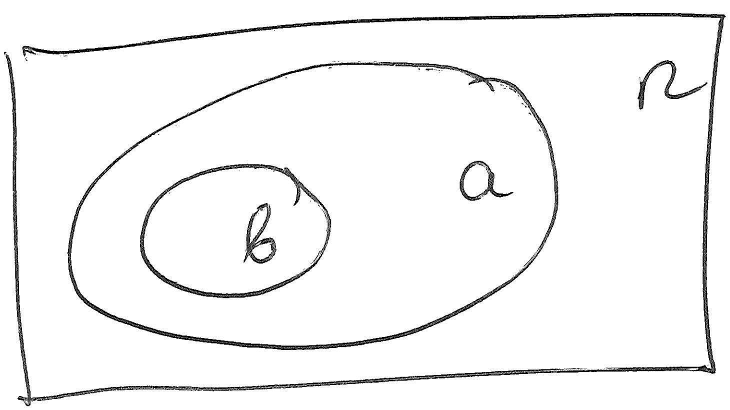
$$= \frac{n(n-1)\cdots(n-a+1)}{b!\,(a-b)!},$$

this yields $\binom{n}{a}\binom{a}{b} = \binom{n}{b}\binom{n-b}{a-b}$. $\square$

<u>Remark.</u> In the case when $n, a, b \in \mathbb{N}$, Prop. 3.2 can be proven bijectively as well:

$\binom{n}{a}\binom{a}{b}$ is the number of ways to choose a committee of a people from a set of n people and then ~~choose~~ chose a subcommittee of b people from this committee.

$\binom{n}{b}\binom{n-b}{a-b}$ is the same number, but we're organizing the choice differently: first, we choose the subcommittee, & then we choose a-b more people to add to it, forming the committee.



Cor. 3.3. ~~For~~ Let $n \in \mathbb{N}$. Then, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = [n=0]$.

(Recall: $[n=0] = \begin{cases} 1, & \text{if } n=0; \\ 0, & \text{if } n \neq 0 \end{cases}$. Equivalently, $[n=0] = \delta_{n,0}$ using the Kronecker delta.)

(Compare with Cor. 1.16b, which says that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.)

Proof of Cor. 3.3. Set $x = -1$ and $y = 1$ in Thm. 1.16a.

Get $((-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cancel{1^{n-k}} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$.

Thus, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = ((-1)+1)^n = 0^n = [n=0]$. $\square$

2nd proof of Cor. 3.3. Rewrite $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$ as

$$\sum_{k \text{ even}} \binom{n}{k} - \sum_{k \text{ odd}} \binom{n}{k}. \quad (\text{with } k \in \{0, \ldots, n\}).$$

Thus, for $n > 0$, we must prove $\sum_{k \text{ even}} \binom{n}{k} - \sum_{k \text{ odd}} \binom{n}{k} = 0$.

In other words, we must prove

$$(2) \qquad \sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k},$$

We can prove (2) bijectively:

The LHS of (2) is the # of subsets of $[n]$ having even size.

The RHS of (2) is the # of subsets of $[n]$ having odd size.

So we need a bijection

$\{\text{subsets of } [n] \text{ having even size}\} \to \{\text{subsets of } [n] \text{ having odd size}\}$.

One bijection we can use here is

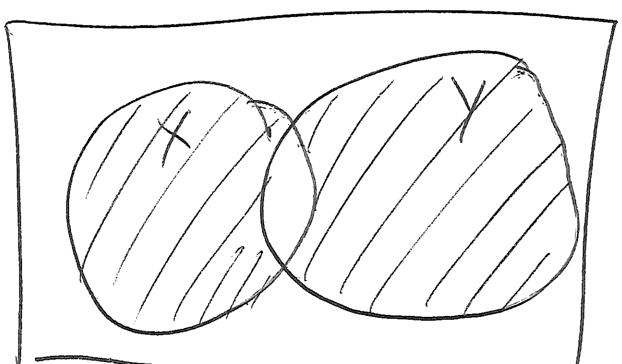
$$A \longmapsto \begin{cases} A \cup \{n\}, & \text{if } n \notin A; \\ A \setminus \{n\}, & \text{if } n \in A \end{cases}$$

(the inverse map is given by the same formula).

This proves Cor. 3.3 for $n>0$. (The $n=0$ case is easy.) $\square$

Rmk. If $X$ and $Y$ are two sets, then the symmetric difference of $X$ and $Y$ is a set called $X \triangle Y$, defined by

$$X \triangle Y = \{\text{all elements lying in exactly one of } X \text{ and } Y\}$$
$$= (X \cup Y) \setminus (X \cap Y) = (X \setminus Y) \cup (Y \setminus X).$$




Note that

$(X \triangle Y) \triangle Z = X \triangle (Y \triangle Z)$,

$X \triangle Y = Y \triangle X$,

$X \triangle X = \emptyset$,

$X \cap (Y \triangle Z) = (X \cap Y) \triangle (X \cap Z)$, ...

2nd proof is thus given

The bijection in the above by $A \longmapsto A \triangle \{n\}$.