Proof of Induction Principle 2.16.

For each $n \in \mathbb{Z}_{\geq g}$, define the statement $\mathcal{B}(n)$ by

$$\mathcal{B}(n) = \left(\mathcal{A}(g) \wedge \mathcal{A}(g+1) \wedge \cdots \wedge \mathcal{A}(n-1)\right)$$

$$= \left(\mathcal{A}(k) \text{ holds } \forall k \in \mathbb{Z}_{\geq g} \text{ with } k<n\right).$$

Now, we want to prove $\mathcal{B}(n)$ $\forall n \in \mathbb{N}$ using Induction principle 2.10.

So we must prove the following:

(1) $\mathcal{B}(g)$ holds.

(2) $\forall n \in \mathbb{Z}_{\geq g}$, if $\mathcal{B}(n)$ holds, then $\mathcal{B}(n+1)$ holds.

Proof of (1): $\mathcal{B}(g) = (\mathcal{A}(k) \text{ holds } \forall \underbrace{k \in \mathbb{Z}_{\geq g} \text{ with } k<g}_{\text{there are no such } k})$

is vacuously true. So it holds.

Proof of (2): Let $n \in \mathbb{Z}_{\geq g}$. Assume that $\mathcal{B}(n)$ holds.

We must prove that $\mathcal{B}(n+1)$ holds.

We know that $\mathcal{B}(n)$ holds, i.e., we have

(3) $\mathcal{A}(k)$ holds $\forall k \in \mathbb{Z}_{\geq g}$ with $k<n$.

In other words,

$$\mathcal{A}(m) \text{ holds } \forall m \in \mathbb{Z}_{\geq g} \text{ with } m < n.$$

So, by our only assumption, $\mathcal{A}(n)$ holds.
Combining this with (3), we conclude that

$$\mathcal{A}(k) \text{ holds } \forall k \in \mathbb{Z}_{\geq g} \text{ with } k < n+1.$$

In other words, $\mathcal{B}(n+1)$ holds. So (2) is proven.
Thus, by Principle 2.10, $\mathcal{B}(n)$ holds $\forall n \in \mathbb{Z}_{\geq g}$.
Therefore, $\mathcal{B}(n+1)$ holds $\forall n \in \mathbb{Z}_{\geq g}$.
Now, let $n \in \mathbb{Z}_{\geq g}$. Then, $\mathcal{B}(n+1)$ holds.
In other words,

$$\mathcal{A}(k) \text{ holds } \forall k \in \mathbb{Z}_{\geq g} \text{ with } k < n+1.$$

Applying this to $k = n$, we get that $\mathcal{A}(n)$ holds.
This yields Induction principle 2.16. $\square$

A source of applications of strong induction are properties of Fibonacci numbers:

$(f_0, f_1, f_2, f_3, \dots)$ is defined by

$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad \forall n \geq 2.$

n	0	1	2	3	4	5	6	7	8
f_n	0	1	1	2	3	5	8	13	21

Lemma 2.17. The Fibonacci sequence $(f_0, f_1, f_2, \dots)$ is strictly increasing from f_2 on: $f_2 < f_3 < f_4 < f_5 < \dots$.

Proof. Use Induction principle 2.16 (strong induction) with $\mathcal{A}(n) = (f_n < f_{n+1})$ and $g = 2$.

We need to prove: $\forall n \in \mathbb{Z}_{\geq 2}$, if ($\mathcal{A}(m)$ holds $\forall m < n$), then $\mathcal{A}(n)$ holds.

So let $n \in \mathbb{Z}_{\geq 2}$. Assume that $\mathcal{A}(m)$ holds $\forall m < n$. This is the induction hypothesis. We want to prove $\mathcal{A}(n)$, in other words, we want to prove $f_n < f_{n+1}$.

This is equivalent to $0 < f_{n-1}$, since $f_{n+1} = f_n + f_{n-1}$.

If $n=2$, this is clear $(0<1=f_{2-1})$.
So WLOG assume $n \neq 2$. Thus, $n \geq 3$.
Now, ind. hypothesis says $A(m)$ holds $\forall m<n$.
In other words, $f_m < f_{m+1}$ holds $\forall m<n$.
In other words, $f_2 < f_3 < \cdots < f_n$.
Thus, $f_2 \leq f_{n-1}$. Thus, $f_{n-1} \geq f_2 = 1 > 0$, so $0 < f_{n-1}$.

So $A(n)$ is proven.
Thus, the induction is complete, i.e., Principle 2.16 yields $A(n)$ $\forall n \geq 2$. In other words, $f_n < f_{n+1}$ $\forall n \geq 2$.
In other words, $f_2 < f_3 < f_4 < \cdots$. $\square$

~~Lemma 2.1~~

Recall: A set S of integers is <u>lacunar</u> if it contains no 2 consecutive integers.

<u>Lemma 2.18.</u> Let S be a finite lacunar subset of $\mathbb{Z}_{\geq 2}$.
Then, $\sum_{t \in S} f_t < f_{\max S + 1}$,

where $\max S$ means the largest element of S (this is defined to be 0 if $S=\emptyset$).

Example. Let $S=\{3,5,8\}$. Then,

$$\sum_{t\in S} f_t = f_3+f_5+f_8 = 2+5+21 = 28$$
$$< f_9 = 34,$$

Proof of Lemma 2.18. Induction on $|S|$.

This means: Let $A(n) = $ (Lemma 2.18 holds for all S of size n).

Apply Induction principle 2.1.

Induction base: Lemma 2.18 holds for $|S|=0$. (Indeed, in this case, $\sum_{t\in S} f_t = 0 < f_1 = f_{\max S+1}$.)

Induction step: Let $m\in\mathbb{N}$. Assume (as the ind. hypothesis) that Lemma 2.18 holds for $|S|=m$.

We must prove that Lemma 2.18 holds for $|S|=m+1$.

So let S be a finite lacunar subset of $\mathbb{Z}_{\geq 2}$ with $|S| = m+1$.

Let $S' = S \setminus \{\max S\}$. This S' has size $|S'| = m$.

So the Ind. hypothesis (applied to S' instead of S) yields

$$(4) \qquad \sum_{t \in S'} f_t < f_{\max S' + 1}.$$

Now,

$$\sum_{t \in S} f_t = f_{\max S} + \sum_{t \in S'} f_t \overset{\text{by (4)}}{<} f_{\max S} + f_{\max S' + 1}.$$

But S is lacunar, so $\max S' + 1 \leq \max S - 1$.

Hence, $f_{\max S' + 1} \leq f_{\max S - 1}$

(since $(f_0, f_1, f_2, \ldots)$ is weakly increasing).

So

$$\sum_{t \in S} f_t < f_{\max S} + \underbrace{f_{\max S' + 1}}_{\leq f_{\max S - 1}} \leq f_{\max S} + f_{\max S - 1} = f_{\max S + 1}.$$

In other words, Lemma 2.18 holds for S.
This completes the induction step. (since we have proven that ~~$\mathcal{A}(n)$ holds~~ $\mathcal{A}(m+1)$ holds).
Thus, $\mathcal{A}(n)$ holds $\forall n \in \mathbb{N}$. Thus, Lemma 2.18 holds. $\square$

Now, recall the binary system:

$$23 = [1\,0\,1\,1\,1]_2 = 2^0 + 2^1 + 2^2 + 2^4.$$

~~We can~~ So every ~~number~~ $n \in \mathbb{N}$ has a unique representation as a sum of distinct powers of 2.
In other words, every $n \in \mathbb{N}$ has a unique representation ("binary representation") as $n = \sum_{t \in S} 2^t$, where S is a finite subset of $\mathbb{N}$.
(E.g., for $n = 23$, this S is $\{0, 1, 2, 4\}$.)

<u>Claim:</u> Every $n \in \mathbb{N}$ has a unique representation as a sum of distinct non-consecutive Fibonacci numbers

("Zeckendorf representation"),

~~Thm.~~ **Def.** A <u>Zeckendorf representation</u> of an $n \in \mathbb{N}$ is a finite lacunar subset S of $\mathbb{Z}_{\geq 2}$ such that

$$n = \sum_{t \in S} f_t .$$

Example.

$$32 = \underbrace{21}_{=f_8} + \underbrace{11}_{= \underbrace{8}_{=f_6} + \underbrace{3}_{= \underbrace{3}_{=f_4} + 0}} = f_8 + f_6 + f_4$$

$$= \sum_{t \in S} f_t \quad \text{with} \quad S = \{4, 6, 8\}.$$

Thm. 2.19 (Zeckendorf). Every $n \in \mathbb{N}$ has a unique Zeckendorf representation.

Remark. We can use ~~the~~ Thm. 2.19 to define a binary operation $*$ on $\mathbb{N}$ as follows:

Given $p, q \in \mathbb{N}$, set

$$p * q = \sum_{\substack{(s,t) \\ \in S \times T}} f_{s+t} ,$$

where S is the Zeckendorf representation of p
and T ——— " ——— q.

This operation $*$ is called <u>Fibonacci multiplication</u>.

Knuth proved that $*$ is associative.
(We won't prove it here.)

<u>Proof of Thm. 2.19.</u> <u>Existence:</u> For each $n \in \mathbb{N}$, let

$\mathcal{A}(n) = $ (n has a Zeckendorf representation),

We'll prove $\mathcal{A}(n)$ by strong induction (i.e., Induction principle 2.16).

So fix $n \in \mathbb{N}$. Assume $\mathcal{A}(m)$ holds $\forall m < n$.
ind. hyp.

We must prove that $\mathcal{A}(n)$ holds.

WLOG assume $n > 0$ (since for $n = 0$, $\mathcal{A}(n)$ is obvious).

Then, there exists a largest $k \in \mathbb{Z}_{\geq 2}$ such that $f_k \leq n$ (since the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$ increases strictly from $f_2 = 1$ on). Fix this k.

Thus, $f_k \leq n$ but $f_{k+1} \not\leq n$, so $n < f_{k+1}$.

Now, $n - f_k \in \mathbb{N}$ (since $f_k \leq n$) and $n - f_k < n$ (since $f_k \geq f_2 = 1 > 0$). Hence, by the ind. hyp., $\mathcal{A}(n - f_k)$ holds. Thus, $n - f_k$ has a Zeckendorf repres.

i.e., ~~$\exists$ subset~~ $\exists$ finite lacunar subset S' of $\mathbb{Z}_{\geq 2}$ such that $n - f_k = \sum_{t \in S'} f_t$. ~~Thus,~~ Consider this S'.

Thus, $n = \sum_{t \in S'} f_t + f_k$.

Observation 1: $k > \max S' + 1$.

[Proof of observation 1: If $S' = \emptyset$, do it yourself.

So WLOG assume $S' \neq \emptyset$.

Thus, $\sum_{t \in S'} f_t \geq f_{\max S'}$.

Hence $n - f_k = \sum_{t \in S'} f_t \geq f_{\max S'}$, so $f_{\max S'} \leq n - f_k$

$< f_{k+1} - f_k$ (since $n < f_{k+1}$).

Thus, $f_{\max S'} < f_{k-1}$ (since $f_{k+1} - f_k = f_{k-1}$).

Hence, $\max S' < k-1$ (since $(f_0, f_1, f_2, \ldots)$ is weakly increasing).

Thus, $k > \max S' + 1$.]

Observation 1 yields that $k \notin S'$ and that $S' \cup \{k\}$ is still lacunar (since S' is lacunar).

Therefore,

$$n = \sum_{t \in S'} f_t + f_k = \sum_{t \in S' \cup \{k\}} f_t$$ is a Zeckendorf representation of n.

Thus, n has a Zeckendorf repr.
In other words, $\mathcal{A}(n)$ holds. Thus, the induction proof is complete (i.e., $\mathcal{A}(n)$ holds $\forall n$).
So we have prove that each $n \in \mathbb{N}$ has a Zeckendorf repr.

<u>Uniqueness:</u> For each $n \in \mathbb{N}$, let

$\mathcal{A}(n) =$ (n has <u>at most</u> one Zeckendorf repr.)
$=$ (any two ZRs of n are equal).

("ZR" means "Zeckendorf representation").

Again, prove this by strong induction.
So fix $n \in \mathbb{N}$. Assume $\mathcal{A}(m)$ holds $\forall m < n$.
We must prove that $\mathcal{A}(n)$ holds.
WLOG assume $n > 0$.
Let S_1 and S_2 be two ZRs of n. We must prove $S_1 = S_2$.

Since S_1 is a ZR of n, we know that S_1 is a finite lacunar subset of $\mathbb{Z}_{\geq 2}$ with $n = \sum_{t \in S_1} f_t$.

Similarly for S_2. Note that $S_1 \neq \emptyset$, since $\sum_{t \in S_1} f_t = n \neq 0$.

Now, Lemma 2.18 (applied to $S = S_1$) yields

$$\sum_{t \in S_1} f_t < f_{\max S_1 + 1},$$

so $$f_{\max S_1 + 1} > \sum_{t \in S_1} f_t = n = \sum_{t \in S_2} f_t \geq f_{\max S_2}.$$

Hence, $\max S_1 + 1 > \max S_2$ (since $(f_0, f_1, f_2, \ldots)$ is weakly increasing).

Therefore, $\max S_1 \geq \max S_2$.

The same argument (with S_1 and S_2 interchanged) yields
$$\max S_2 \geq \max S_1.$$

Thus, $$\max S_1 = \max S_2.$$

Let $m = \max S_1 = \max S_2$.

Then, $S_1 \setminus \{m\}$ and $S_2 \setminus \{m\}$ are two ZRs of $n - f_m$.

Moreover, $n - f_m \in \mathbb{N}$ (since it has a ZR), and $n - f_m < n$

~~Thus,~~ (since $f_m > 0$). Thus, the ind. hyp. yields that

$\mathcal{A}(n - f_m)$ holds, i.e. any two ZRs of $n - f_m$ are equal.

So $S_1 \setminus \{m\} = S_2 \setminus \{m\}$. ~~Combining~~

Now, $S_1 = \underbrace{(S_1 \setminus \{m\})}_{= S_2 \setminus \{m\}} \cup \{m\} = (S_2 \setminus \{m\}) \cup \{m\} = S_2$.

So we have shown that any two ZRs of n are equal.

In other words, $\mathcal{A}(n)$ holds, $\Rightarrow$ Induction complete.

$\square$

So Thm. 2.19 is proven.

__Cor. 2.20.__ Let $p \in \mathbb{Z}_{\geq 2}$. Then, the map

$$\{\text{lacunar subsets of } \{2, \ldots, p\}\} \to \{0, 1, \ldots, f_{p+1} - 1\},$$

$$S \mapsto \sum_{t \in S} f_t$$

is a bijection.

Proof. The map is

- well-defined (by Lemma 2.18:
$$\sum_{t \in S} f_t < f_{\underbrace{\max S}_{\leq p} + 1} \leqslant f_{p+1},$$
so $\sum_{t \in S} f_t \in \{0, 1, \ldots, f_{p+1} - 1\}$).

- injective (since ~~the~~ each n has a unique Zeckendorf repr.).

- surjective (indeed, let $n \in \{0, 1, \ldots, f_{p+1} - 1\}$; then, the Zeckendorf representation S of n satisfies
$$f_{p+1} > n = \sum_{t \in S} f_t \geqslant \cancel{\max} f_{\max S},$$
so $p+1 > \max S$, so $\max S < p$,
so $S \subseteq \{2, \ldots, p\}$;
so our map sends this S to n).

So it is a bijection, $\square$

Recall Prop. 1.22: It says that $\forall n \in \mathbb{N}$, the # of lacunar subsets of $[n]$ is f_{n+2}.

<u>3rd proof of Prop. 1.22:</u> WLOG $n > 0$. Set $p = n+1$. Thus, $p \in \mathbb{Z}_{\geq 2}$. Hence, Cor. 2.20 yields

$$|\{\text{lacunar subsets of } \{2,\ldots,p\}\}| = |\{0,1,\ldots,f_{p+1}-1\}|$$
$$(1) \qquad = f_{p+1} = f_{n+2} \quad (\text{since } p+1 = n+2).$$

But

$$\{\text{lacunar subsets of } \{2,\ldots,p\}\} \to \{\text{lacunar subsets of } [n]\},$$
$$S \mapsto \{s-1 \mid s \in S\}$$

is a bijection. Thus,

$$|\{\text{lacunar subsets of } \{2,\ldots,p\}\}| = |\{\text{lacunar subsets of } [n]\}|.$$

Comparing this with (1), we get

$$|\{\text{lacunar subsets of } [n]\}| = f_{n+2},$$

qed. □