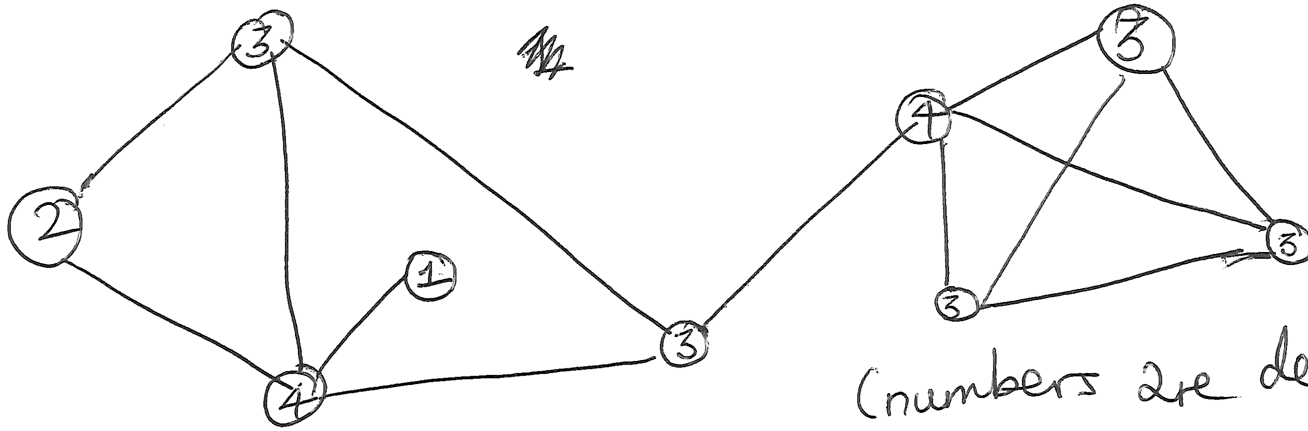
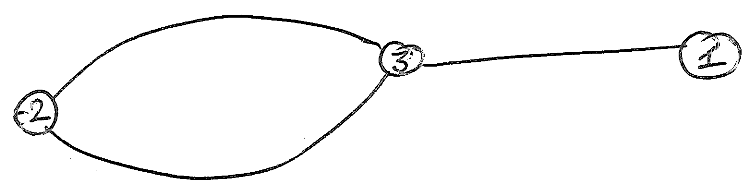
Prop. 6.4. Let $G=(V,E)$ be a simple graph, with ≥ 2 vertices. Then, G has 2 vertices of equal degree.

Example:



(numbers are degrees).

Non-example:



Proof of Prop. 6.4. Assume the contrary. Let $n=|V|$. So $n\geq 2$. Thus, the map $\deg : V \to \{0,1,\ldots,n-1\}$ is well-defined (since G is a simple graph) and injective (by assumption).

Hence, this map is bijective (by the pigeonhole principle for injections).

Hence, $\exists$ vertex p with $\deg p = 0$

and $\exists$ vertex q with $\deg q = n-1$.

Note $p \neq q$ (since $\deg p = 0 \neq n-1 = \deg q$).

Thus, q is adjacent to each other vertex (since $\deg q = n-1$), in particular to p. Hence $\deg p \geq 1$

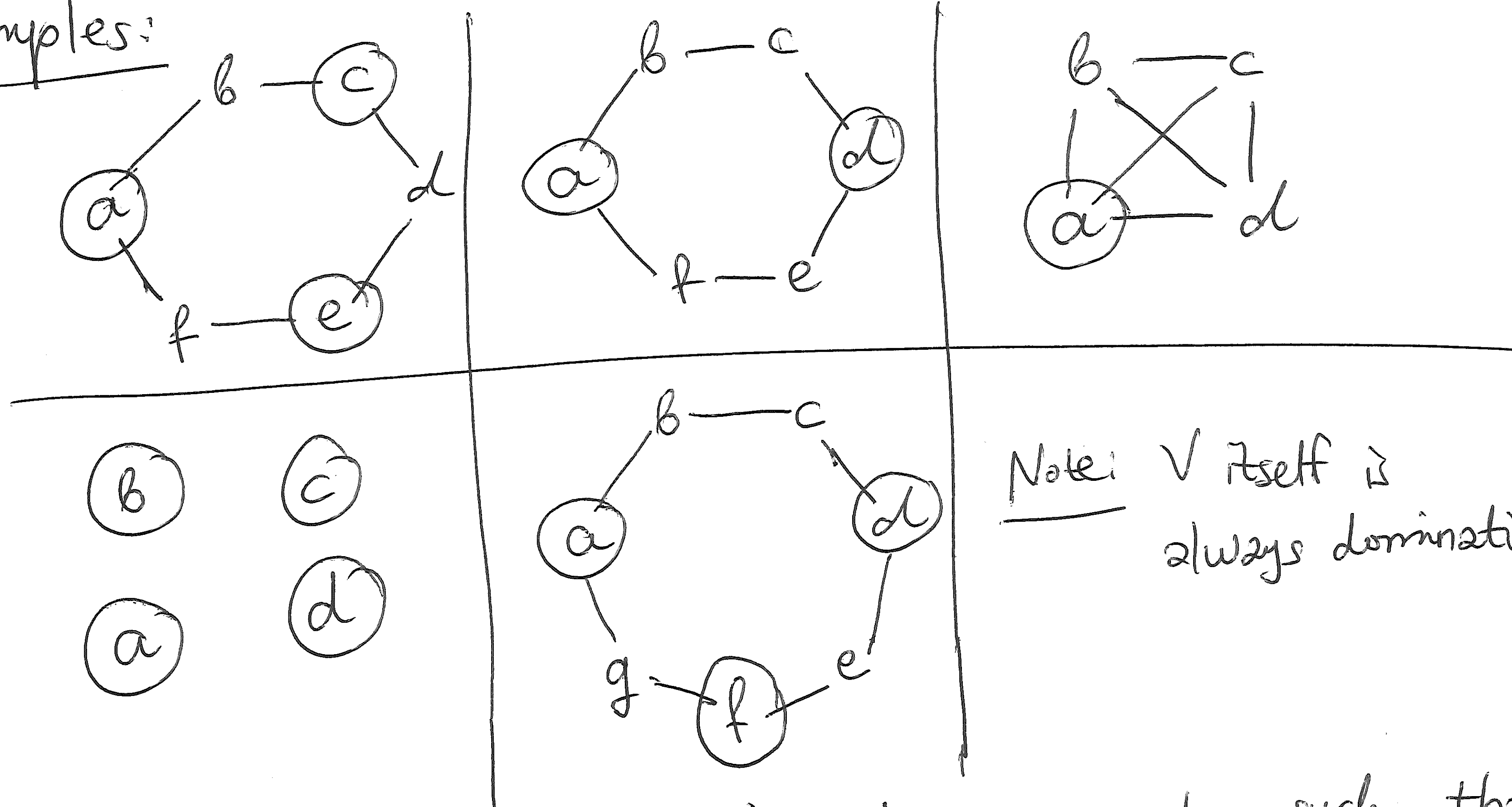
$\lightning$ $\deg p = 0$.

$\square$

6.3. DOMINATING SETS

Def. A *dominating set* of a graph $G = (V, E, \varphi)$ is a subset S of V such that each vertex not in S has ≥ 1 neighbor in S.

Examples:



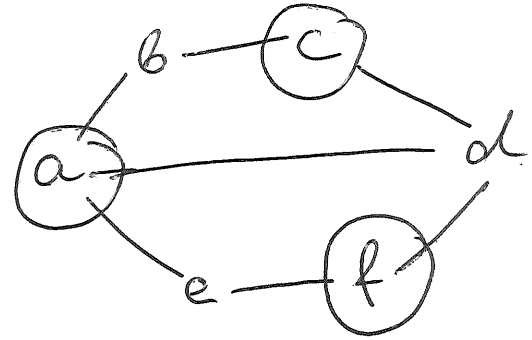
Note: V itself is always dominating!

Prop. 6.5. Let $G = (V, E, \varphi)$ be a graph such that each vertex v has $\deg v \geq 1$.

(a) There are two disjoint dominating sets of G with union of V.

(b) $\exists$ dominating set of size $\leq |V|/2$.

Proof (a) Define an independent set of G to be a ~~subset~~ set of mutually non-adjacent vertices. (Example:



)

Pick a maximum-size independent set A of G. Let $B = V \setminus A$.

Claim: A and B are disjoint & dominating.

(Indeed, any $b \in B$ is adjacent to some elt. of A, or else it could be added to A. $\Rightarrow$ A is dominating.

Also, any $a \in A$ has at least 1 neighbor, and this neighbor must be in B (since A is independent). $\Rightarrow$ B is dominating.)

Of course, $A \cup B = V$.

(b) follows from (a). $\square$

Theorem 6. (Brouwer 2009). Let G be a graph. Then, the # of dominating sets of G is odd.

See [Inogra (Spring 2017 Math 5707 notes), Ch. 3].

Note that simple graphs are sufficient, since parallel edges can be ignored (i.e, we can replace a graph (V, E, φ) by the simple graph $(V, \varphi(E))$).

6.4. PATHS, WALKS, CONNECTIVITY

Def. Let $G = (V, E, \varphi)$ be a graph. Let $p, q \in V$.

(a) A <u>walk</u> from p to q means a list $(v_0, e_1, v_1, e_2, v_2, \ldots\ldots, v_{k-1}, e_k, v_k)$, where $v_i \in V$, $e_i \in E$, $\varphi(e_i) = \{v_{i-1}, v_i\}$, $v_0 = p$, $v_k = q$.

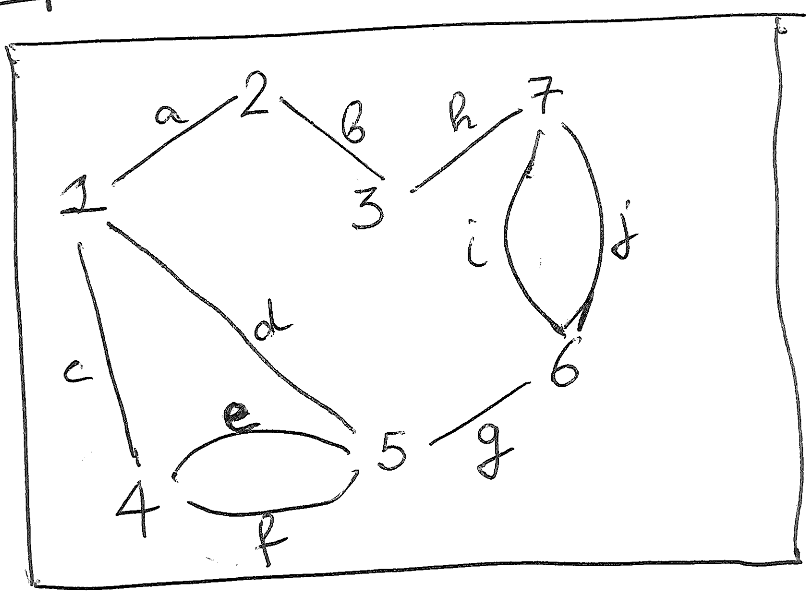
We shall abbreviate "walk from p to q" as "walk $p \to q$".

(b) A walk like this is called a <u>path</u> if all the v_i are distinct.

(c) A walk from p to q is called a circuit (or a closed walk) if $p=q$.

(d) A walk $p \to q$ is called a cycle if $p=q$, but $v_0, v_1, \dots, v_{k-1}$ are distinct, $e_1, e_2, \dots, e_k$ are distinct, and $k \geq 1$;

Example:



$(1, a, 2, b, 3, h, 7, i, 6, j, 7)$ is a walk $1 \to 7$, but not a path.

$(1, c, 4, f, 5)$ is a path $1 \to 5$.

$(1, c, 4, e, 5)$ is another path $1 \to 5$.

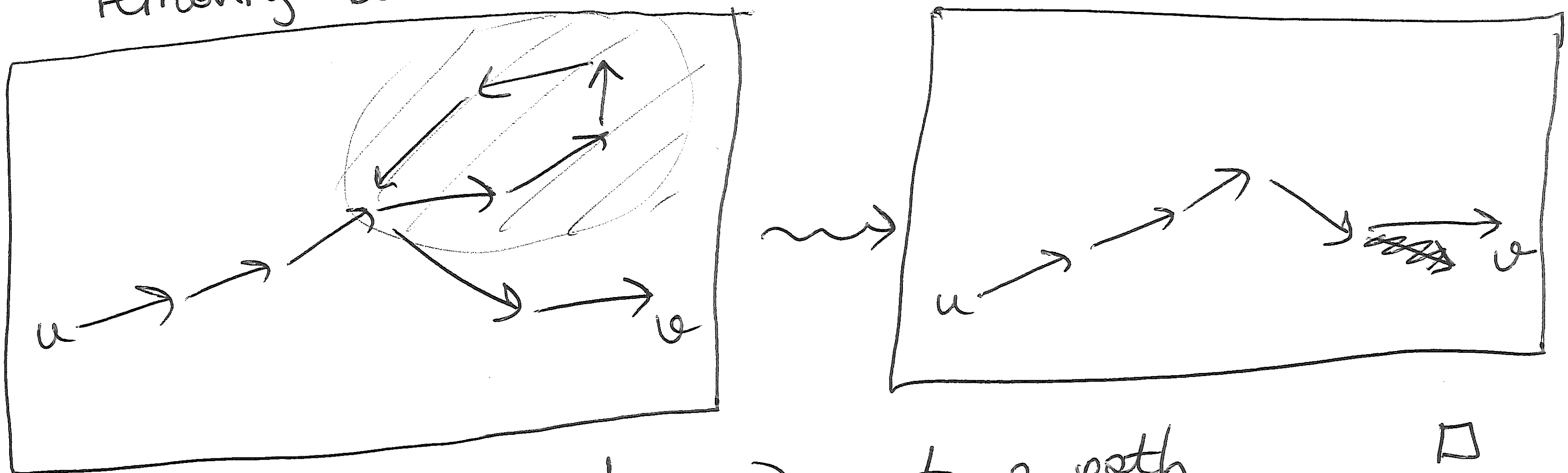
$(4, e, 5, f, 4)$ is a cycle and thus a circuit.

$(4, e, 5, e, 4)$ is a circuit, but NOT a cycle.

(3) is a circuit and a path $3 \to 3$, but not a cycle.

Prop. 6.7. Let u, v be two vertices of a graph G.
If $\exists$ a walk $u \to v$, then $\exists$ a path $u \to v$.

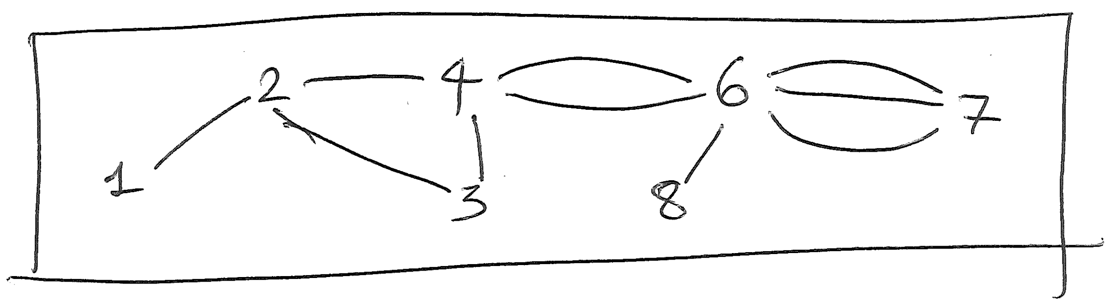
Proof. A walk that is not a path can be shortened by removing a circuit.



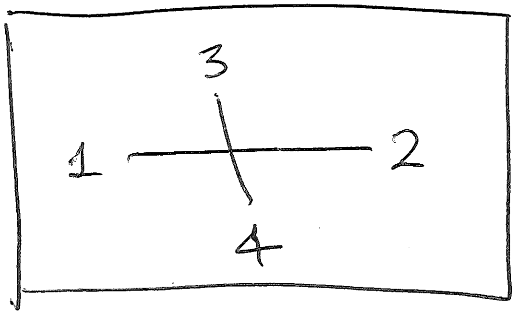
Do this recursively $\Rightarrow$ get a path. $\square$

Def. A graph G is connected if G has ≥ 1 vertex, and $\forall$ vertices u and v, $\exists$ walk $u \to v$.

Examples:



is connected.



is not connected.

Def. Let $G = (V, E, \varphi)$ be a graph.

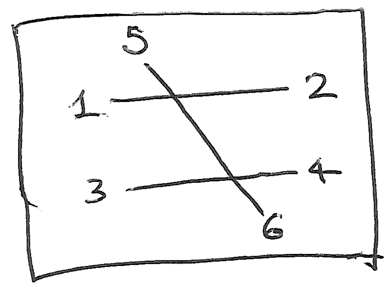
(a) We define a relation $\overset{G}{\sim}$ on V as follows:

$$v \overset{G}{\sim} w \iff \exists \text{ walk } v \to w.$$

This $\overset{G}{\sim}$ is an equivalence relation.

(b) The equivalence classes of $\overset{G}{\sim}$ are the called the **connected components** of G.

Example: ~~Sometime~~ The connected components of



are $\{1,2\}$, $\{3,4\}$ and $\{5,6\}$.

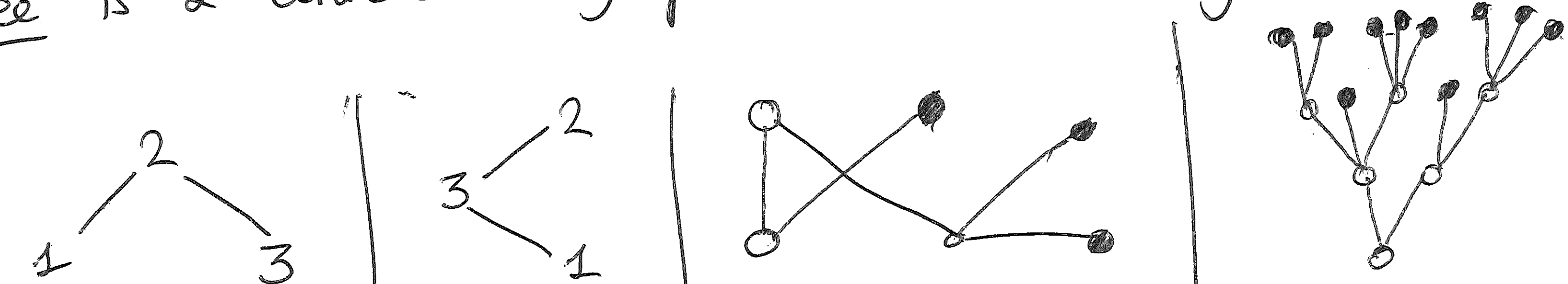
Remark: A graph is connected if and only if its number of connected components is 1.

6.5. TREES

Def. A *leaf* of a graph G is a vertex v with $\deg v = 1$.

Def. A *tree* is a connected graph that has no cycles.

Examples:



leaves: 1, 3	leaves: 1, 2	leaves: •	leaves: •

Thm. 6.8. (*tree equivalence theorem*). Let $G = (V, E, \varphi)$ be a graph. TFAE (= "the following (statements) are equivalent):

T_1: G is a tree.

T_2: $V \neq \emptyset$, and $\forall u \in V$ and $v \in V$ $\exists$ *unique* path $u \to v$.

T_3: $V \neq \emptyset$, and $\forall u \in V$ and $v \in V$
$\exists$ unique backtrack-free walk $u \to v$
(= walk $u \to v$ with no two consecutive edges identical).

T_4: G is connected, and $|E| = |V| - 1$.

T_5: G is connected, and $|E| < |V|$.

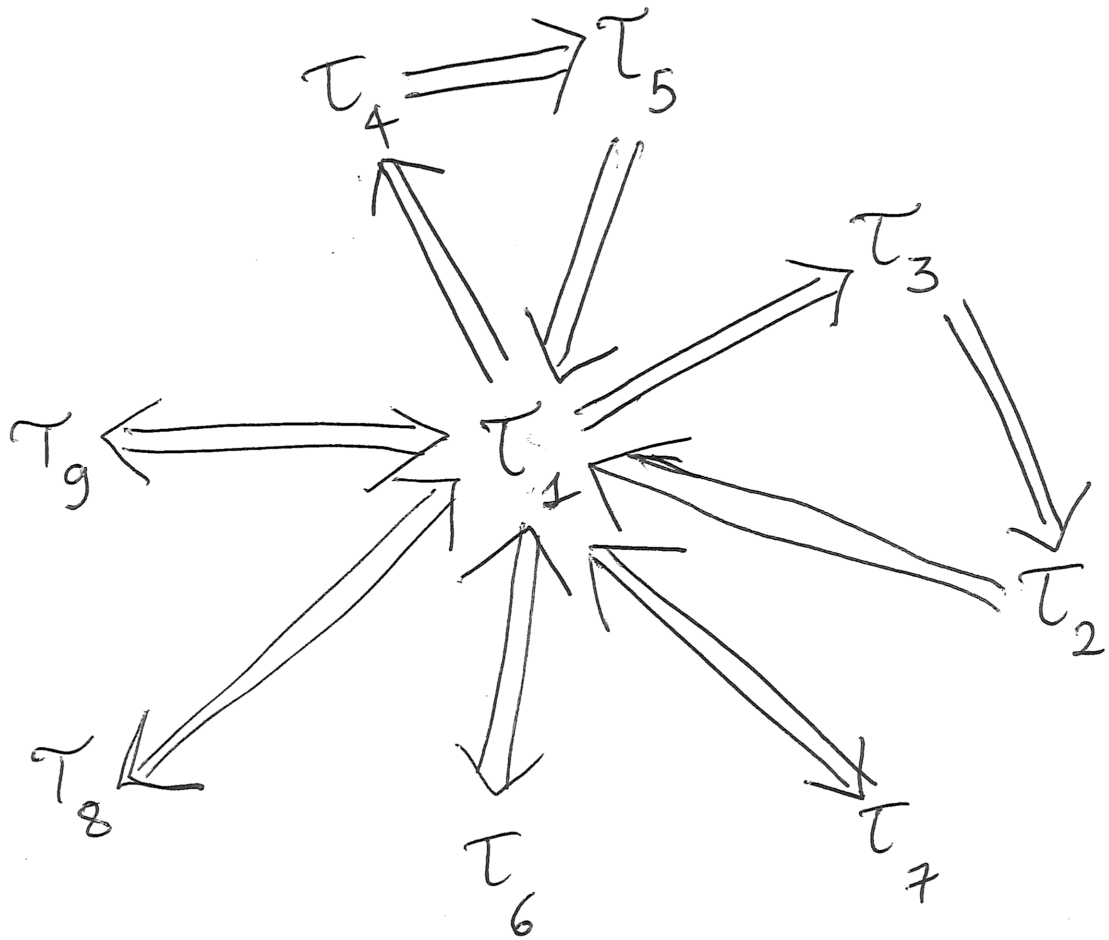
T_6: G has no cycles, but adding any new edge produces a cycle.

T_7: G is connected, but removing any edge disconnects it.

T_8: EITHER $|V| = 1$, OR $\exists$ leaf $v \in V$ such that removing v from V (along with the edge that contains v) produces a tree.

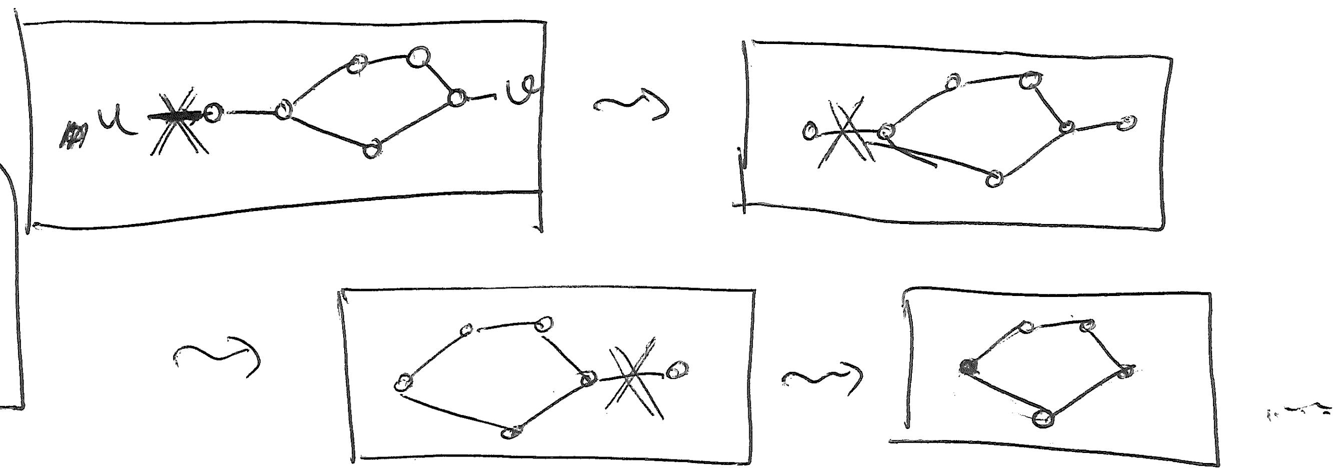
T_9: G has no cycles, and $|E| \geq |V| - 1$, and $V \neq \emptyset$.

Proof outline. For details, see Spring 2017 Math 5707 (proof of Thm. 13).



Observation 1: Let u, v be 2 vertices of G.
If $\exists$ two distinct backtrack-free walks $u \to v$, then G has a cycle.

(Proof idea:

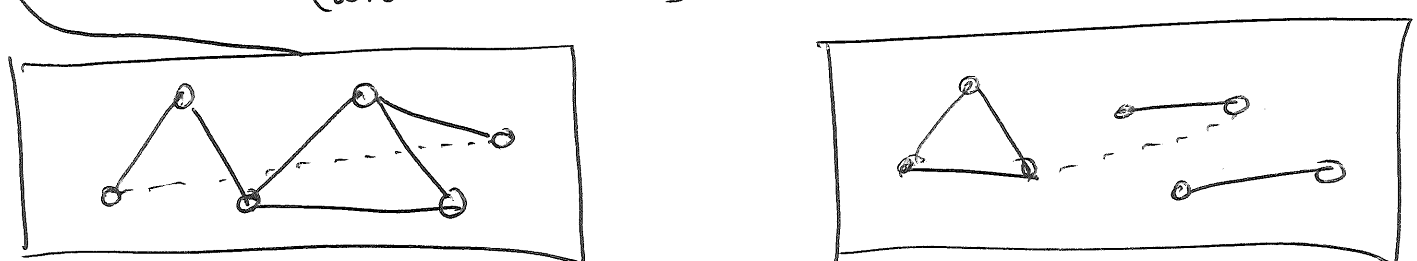


without changing the connected components

...)

Observation 2: Adding an edge to a graph either creates a cycle or decreases the # of connected components by 1 (by merging two of the components) (without creating cycles).

(Examples:



create cycle

merge components

)

Observation 3: Let $G = (V, E, \varphi)$ be a graph with $|E| < |V|$. Then, G has a vertex of degree ≤ 1.

(**Proof:** The handshake lemma says $\sum_{v \in V} \deg v = 2\underbrace{|E|}_{<|V|} < 2|V|$.

Hence, $\exists\, v \in V$ with $\deg v < 2$. In other words, $\exists\, v \in V$ with $\deg v \leq 1$.)

Observation 3 implies that every tree with ≥ 2 vertices has a leaf. $\square$

How can we count trees? Two questions:

1. trees on a given vertex set (regarded as simple graphs).

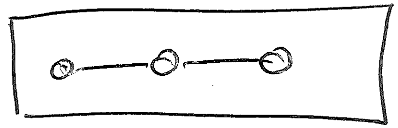
For example, the trees with vertex set $[3]$ are

1 — 2 — 3	2 — 1 — 3	1 — 3 — 2

so there are 3 of them.

n	# of trees
1	1
2	1
3	3
4	16
5	125

2. trees "up to isomorphism" (i.e., up to relabelling vertices).



Question 2 is hard. The best thing we have is asymptotic formula: $\sim \beta \alpha^n n^{-5/2}$, for $\alpha \approx 2.955$, $\beta \approx 0.5349$.

Question 1 has a nice answer: n^{n-2} (for an n-element set). This is Cayley's formula. Next time, proof outline.