8. THE STABLE MARRIAGE PROBLEM

(This is not graph theory, and has nothing to do with Hall's marriage theorem). We follow [LeLeMe, §6.4] (highly recommended).

Suppose we have n men & n women.
Each man ranks all women in order of preference.
Each woman ranks all men —— // ——

We're looking for a way to marry everyone off (men to women).

We want to avoid **rogue couples**: A **rogue couple** in a matching is a pair (m, w), where m is a man & w is a woman & m prefers w to his wife & w prefers m to her husband.

A **stable matching** is a way to marry everyone off such that no rogue couples exist.

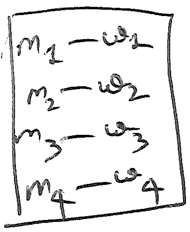
Stable marriage problem: Find a stable matching.

Yes, it always exists! This will follow from the algorithm below.

Example: Let $n = 4$, and the preferences be

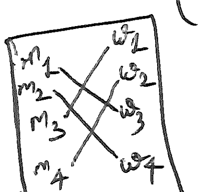
men, prefs:	women
m_1: $1>2>3>4$	w_1: $4>3>1>2$
m_2: $1>4>3>2$	w_2: $2>4>1>3$
m_3: $2>1>3>4$	w_3: $4>1>2>3$
m_4: $4>2>3>1$	w_4: $3>2>1>4$

("$m_2: 1>4>3>2$" means "man m_2 prefers $w_1 > w_4 > w_3 > w_2$").



(a) The matching $\{m_1 w_1, m_2 w_2, m_3 w_3, m_4 w_4\}$ is not stable, since (m_2, w_3) is a rogue couple.

(b) The matching $\{m_1 w_3, m_2 w_4, m_3 w_1, m_4 w_2\}$ is stable.



The <u>Mating Ritual</u> (aka the Gale-Shapley algorithm, or the deferred acceptance algorithm), is an algorithm that finds a stable matching. It proceeds in several "rounds", which we imagine to be several days. On day 1, each man has his full preference list of all n women, & each woman has her full —— // —————— men.

Each day, the following happens:

<u>Morning:</u> Each man proposes to the woman on top of his list. He is said to be her <u>suitor</u>. (Or he stays home, if his list is empty.)

<u>Afternoon:</u> Every woman that has ≥ 1 suitor sends away all her suitors except for her favorite among them.

<u>Evening:</u> Every suitor who got sent away by a woman crosses her off his list.

<u>Termination condition:</u> If, on some day, each woman has ≤ 1 suitor, then each woman marries her suitor (if he

exists), & the algorithm stops.

Example: In the example above:

On day 1: m_1, m_2 propose to w_1; m_3 to w_2; m_4 to w_4.
m_2 is sent away by w_1 & crosses her off his list.

On day 2: m_1 proposes to w_1; m_2, m_4 to w_4; m_3 to w_2.
m_4 is sent away by w_4 & crosses her off his list.

On day 3: m_1 proposes to w_1; m_2 to w_4; m_3, m_4 to w_2,
m_3 is sent away by w_2 & crosses her off his list.

On day 4: m_1, m_3 propose to w_1; m_2 to w_4; m_4 to w_2.
m_1 is sent away by w_1 & crosses her off his list.

On day 5: m_1, m_4 propose to w_2; m_2 to w_4; m_3 to w_1.
m_1 is sent away by w_2 & crosses her off his list.

On day 6: m_1 proposes to w_3; m_2 to w_4; m_3 to w_2; m_4 to w_2.
Each woman has ≤ 1 suitor, so they marry.

Why does the algorithm work?

<u>Lem. 8.1.</u> Let m be a man. Then, the woman he proposes to at day $d+1$ is always <u>at most as</u> peferrable to him as the woman he proposes to at day d.
(In other words, he "works down his list".)

<u>Proof.</u> He always proposes to the best woman on his list, and can only cross her off. □

<u>Lem. 8.2.</u> Let w be a woman. Then, her ~~suitor~~ best suitor at day $d+1$ is always <u>at least as</u> peferrable to her as any of her suitors at day d.

<u>Proof.</u> Her best suitor at day d is still around to propose to her on day $d+1$. □

<u>Lem. 8.3.</u> The algorithm terminates.

<u>Proof.</u> Every day before termination, at least 1 man crosses a woman off his list. So the lists altogether get shorter. $\Rightarrow$ After $\leq n^2$ days, these lists will be empty. □

Lem. 8.4. Let d be any day. Let m be a man, and w be a woman such that w is ~~not on m's list on day d~~ crossed off m's list on day d or earlier. Then, on day d, she has a suitor whom she prefers to m.

Proof. Woman w only sends m away when she has a better suitor. From that point on, she only gets better and better suitors (by Lem. 8.2). $\square$

Lem. 8.5. When the algorithm terminates, everyone is married.

Proof. Assume the contrary. Thus, some man m is unmarried (by the pigeonhole principle). Hence, his list is empty at day d, where d is the day in which the algorithm terminates. Thus, m must have crossed off each woman at day d. Thus, by Lem. 8.4, each woman has a suitor whom she prefers to m. $\Rightarrow$ Each woman has a suitor on day d. $\Rightarrow$ Each woman is married. $\Rightarrow$ By the pigeonhole principle, each men is married (since # of men = # of women). $\square$

Thm. 8.6. The matching produced by the algorithm is a stable matching.

Proof. Assume the contrary. Thus, there is a rogue couple (m, w). Thus, m prefers w to his wife, & w prefers m to her husband. Consider two cases:

Case 1: w is not on m's list when the algorithm terminates. Then, Lemma 8.4 shows that w has a suitor (on the last day) whom she prefers to m. That suitor must be her husband. So she prefers her husband to m, not m to her husband. Contradiction.

Case 2: w is still on m's list when the algorithm terminates. But the highest woman on his list is his wife (because he marries the woman he proposes to on the termination day). Thus, since his wife is not w, we see that he prefers his wife to w, not w to his wife. Contradiction.

So we always have a contradiction. □

Thm. 8,6 states that the Mating Ritual produces a stable matching. Often, there are several. Which one is better for the men; which one for the women?

~~Def. A person p is a feasible spouse for a person~~

Def. A stable matching M is male-prefered to a stable matching N if for every man m, his M-partner is preferrable (or equal) to his N-partner.

Similarly defined female-prefered.

How does the ~~algorithm~~ men fare under the Mating Ritual?

Def. Let p and q be two persons. We say that q is a feasible spouse for p if ∃ a stable matching in which p is married to q.

<u>Lem. 8.9:</u> Let m be a man & w a woman.
Let d be any day.
If w is crossed off m's list at day d, then w is not a feasible spouse of m.

(Roughly speaking: "Men don't lose any options by crossing off women".)

<u>Proof.</u> Strong induction on d:

<u>Induction step:</u> Assume Lem. 8.9 holds for all days $< d$.
Let m be a man & w a woman such that m crosses w off his list on day d. We want to prove that w is not a feasible spouse of m.

Lem. 8.4 says that w has a suitor m' who she prefers to m. Moreover, by the induction hypothesis, any ~~suitor that m~~ woman that m' has crossed off prior to day d was not a feasible spouse of m'. Thus, m' prefers w to all his other feasible spouses.

Now, if m and w were married to each other in some stable matching, then (m', w) would be a rogue couple (since m' is not married to w), which is absurd.
Hence, m & w are not married to each other in any stable matching. $\Longrightarrow$ w is not a feasible spouse for m. $\square$

Thm. 8.10. The Mating Ritual marries each man to his favorite among his feasible wives.

Proof. Follows from Lem. 8.9. $\square$

Lem. 8.11. Let m be a man. Let w be ~~the fav~~ his favorite among his feasible wives. Then, m is the least favorite among w's feasible husbands.

Proof. Assume the contrary. Thus, w has a feasible husband m' that she likes less than m. Hence $\exists$ stable matching M where m' marries w. Now, (m, w) are a rogue couple for M $\Rightarrow$ contradiction. $\square$

Cor. 8.12. The Mating Ritual marries each woman to her least favorite among her feasible husbands. $\square$

See also

[Roth & Sotomayor: Two-sided matching].