7. DIGRAPHS | 7.1. BASICS

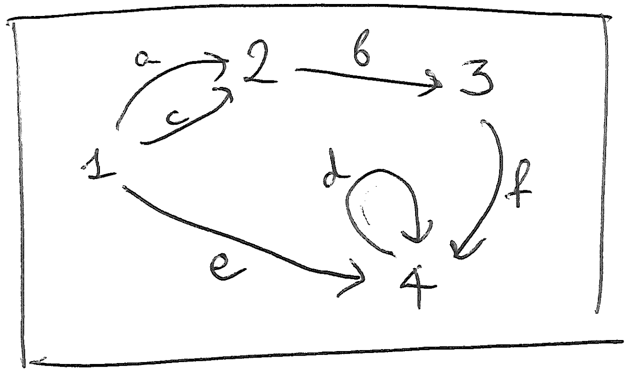
Def. A multidigraph is a triple (V, A, φ), where V and A are finite sets, and $\varphi : A \to V \times V$.

The vertices of this multidigraph (V, A, φ) are the elements of V.

The arcs — // — elements of A.

If $a \in A$ with $\varphi(a) = (u, v)$, then u is the source of a, and v is the target of a. We then draw a as $u \xrightarrow{a} v$.

Remark: "Loops" (i.e., arcs a with $\varphi(a) = (v, v)$) are allowed, e.g.:



We abbreviate "multidigraph" as "digraph".

Vic, in his class, explained Eulerian circuits & oriented spanning trees, (= arborescences).

<u>Def.</u> Let $D = (V, A, \varphi)$ be a multidigraph. Let ~~#~~ $p, q \in V$.

(a) A <u>walk</u> from p to q in D means a list $(v_0, a_1, v_1, a_2, v_2, \ldots, v_{k-1}, a_k, v_k)$, where $v_i \in V$, $a_i \in A$, $\varphi(a_i) = (v_{i-1}, v_i)$, $v_0 = p$, and $v_k = q$.

(b) This walk is called a <u>path</u> if $v_0, v_1, \ldots, v_k$ are distinct.

(c) A walk from p to q is called a <u>circuit</u> if $p = q$.

(d) A circuit $(v_0, a_1, v_1, a_2, v_2, \ldots, v_{k-1}, a_k, v_k)$ is a <u>cycle</u> if $v_0, v_1, \ldots, v_{k-1}$ are distinct and $k \geq 1$.
(Note: The arcs $a_1, \ldots, a_k$ are then distinct automatically.)

Prop. 7.1. Let u and v be two vertices of a multidigraph. If $\exists$ a walk $u \to v$, then $\exists$ path $u \to v$.

Proof. Analogous to Prop. 6.7. $\square$

7.2. Flows & cuts

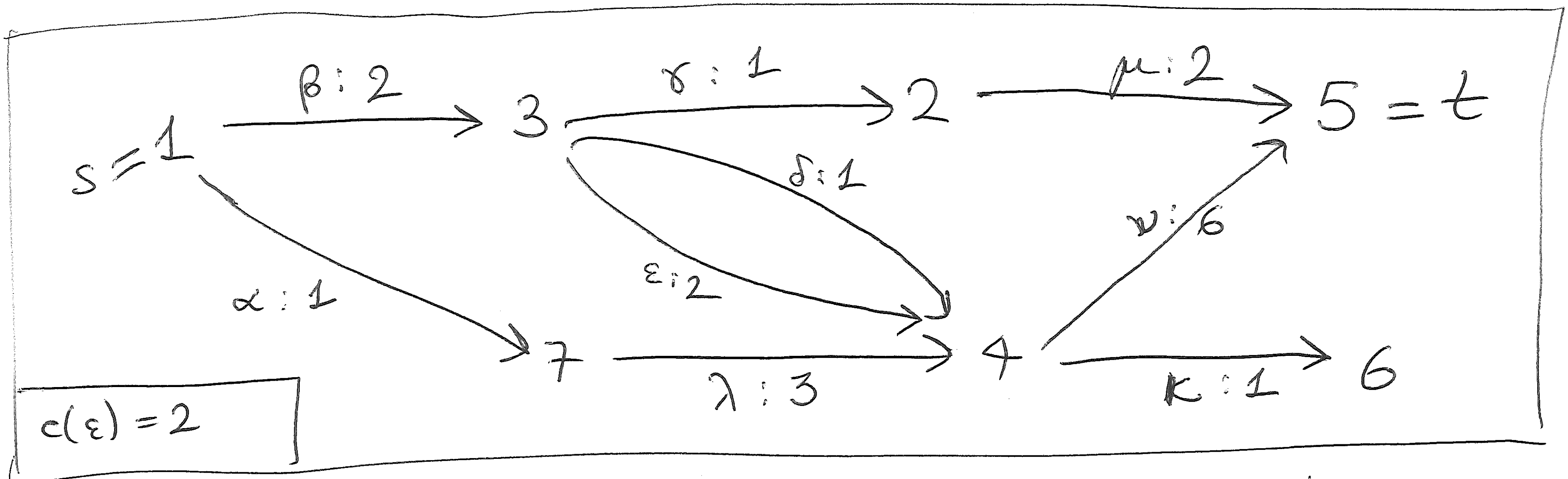
Reference: notes that I'm writing.
(Spring 2017 Math 5707 Lec 16, but with changes.)

Def. A network consits of:

- a multidigraph $D = (V, A, \varphi)$;
- two distinct vertices $s \in V$ and $t \in V$, which we will call the source and the sink. (Don't confuse "the source" with the source of an arc.)
- a function $c : A \to \mathbb{Q}_+$ (where $\mathbb{Q}_+ = \{x \in \mathbb{Q} \mid x \geq 0\}$) called the capacity function.

To draw a network, ~~we~~ we draw D, we mark s and t, and we label each arc a with "$a : c(a)$".

For example, here is a network:



<u>Definition.</u> Let V, A, φ, D, s, t and c be as above.

(a) For any arc $a \in A$, we call $c(a)$ the <u>capacity</u> of a.

(b) For any subset S of V, we let $\overline{S}$ denote the complement $V \setminus S$ of S.

(c) If P and Q are two subsets of V, then $[P, Q]$ will mean the set of all arcs $a \in A$ whose source lies in P and whose target lies in Q.

(d) If P and Q are two subsets of V, ~~then~~ and if $d : A \to \mathbb{Q}_+$ is any function, then we define a number $d(P,Q)$ ~~by~~ $\in \mathbb{Q}_+$ by

$$d(P,Q) = \sum_{a \in [P,Q]} d(a).$$

~~Def~~ Let V, A, φ, D, s, t and c be as above. (for the rest of this section)

<u>Def.</u> A <u>flow</u> (on our network) is a function $f : A \to \mathbb{Q}_+$ with the following properties:

(a) We have $0 \leq f(a) \leq c(a)$ for each $a \in A$. This is called the <u>capacity constraints</u>,

(b) For any vertex $v \in V \setminus \{s,t\}$, we have

$$f^-(v) = f^+(v),$$

where

$$f^-(v) = \sum_{\substack{a \text{ is an} \\ \text{arc with} \\ \text{target } v}} f(a) = f(V, \{v\})$$

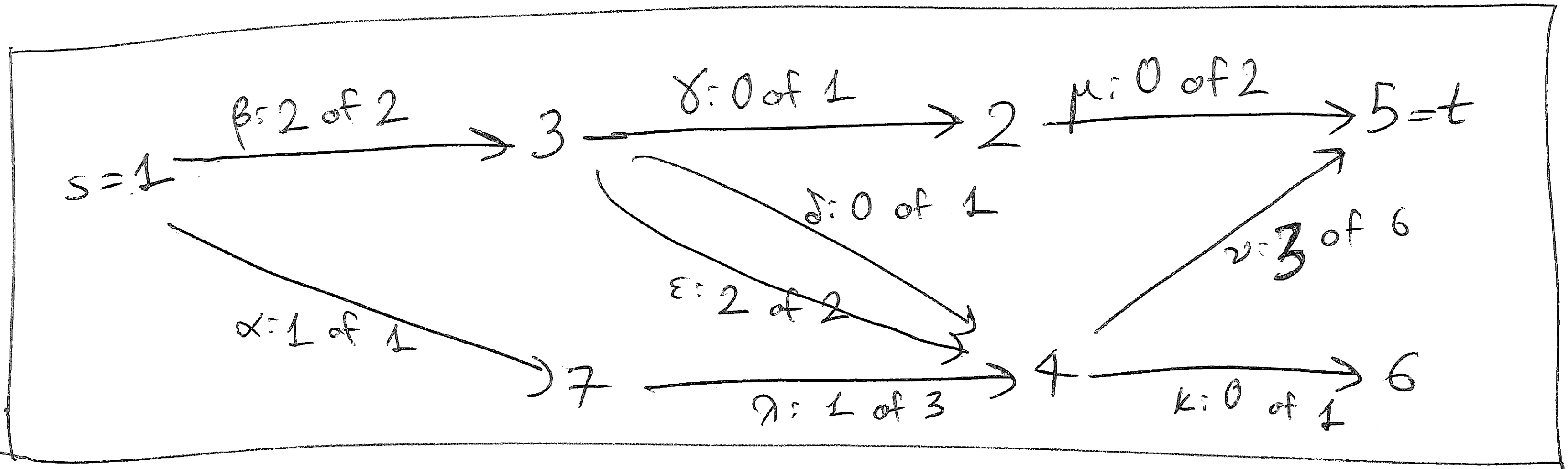
and

$$f^+(v) = \sum_{\substack{a \text{ is an} \\ \text{arc with} \\ \text{source } v}} f(a) = f(\{v\}, V).$$

This is called the <u>conservation constraints</u>.

We draw a flow f on a network in the same way as we draw the network, but we label arc a by "$a : f(a)$ of $c(a)$" instead of "$a : c(a)$".

<u>Example</u> of a flow:



<u>Def.</u> The <u>value</u> of a flow f is defined to be the number $f^+(s) - f^-(s)$. It is denoted by $|f|$.

<u>Prop. 7.2.</u> For any flow f, we have

$$|f| = f^+(s) - f^-(s) = f^-(t) - f^+(t).$$

<u>Proof.</u> We only need to prove the 2nd equality sign.

Each arc $a \in A$ has exactly 1 source. Thus,

$$\sum_{a \in A} f(a) = \sum_{v \in V} \underbrace{\sum_{\substack{a \in A \\ \text{has source } v}} f(a)}_{= f^+(v)} = \sum_{v \in V} f^+(v).$$

Similarly, $\sum_{a \in A} f(a) = \sum_{v \in V} f^-(v)$.

Comparing these, we get $\sum_{v \in V} f^+(v) = \sum_{v \in V} f^-(v)$.

In other words,

$$0 = \sum_{v \in V} f^+(v) - \sum_{v \in V} f^-(v)$$

$$= \sum_{v \in V} \left(f^+(v) - f^-(v)\right)$$

$$= \left(f^+(s) - f^-(s)\right) + \left(f^+(t) - f^-(t)\right) + \sum_{v \in V \setminus \{s,t\}} \underbrace{\left(f^+(v) - f^-(v)\right)}_{=0}$$

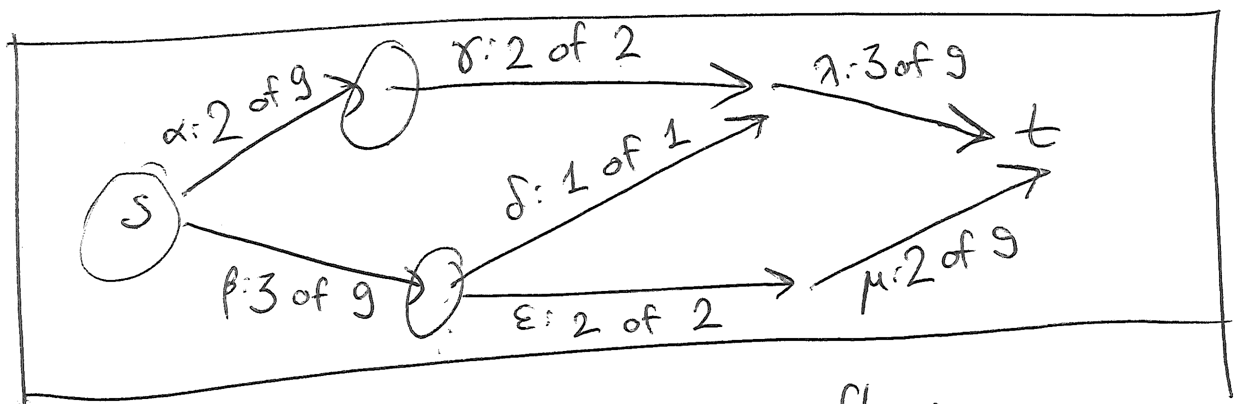
(by conservation constraint)

$$= \left(f^+(s) - f^-(s)\right) + \left(f^+(t) - f^-(t)\right).$$

Thus, $f^+(s)-f^-(s) = f^-(t)-f^+(t)$. $\square$

Problem: Given a network, how to find a maximum flow (i.e., a flow f with maximum $|f|$) ?

Example:



This is a maximum flow.

Prop. 7.3. Let $f: A \to \mathbb{Q}_+$ be a flow. Let $S \subseteq V$.

(a) We have $f(S,\bar{S}) - f(\bar{S},S) = \sum_{v\in S} \left(f^+(v) - f^-(v)\right)$.

(b) If $s \in S$ and $t \notin S$, then $|f| = f(S,\bar{S}) - f(\bar{S},S)$.

(c) If $s \in S$ and $t \notin S$, then $|f| \leq c(S,\bar{S})$.

$c(S,\bar{S})$ is called the capacity of the cut $[S,\bar{S}]$.

So Prop. 7.3 (c) says that the value of any flow is $\leq$ the capacity of any cut.

Thm. 7.4. (Maximum-flow-minimum-cut theorem, short MFMC) (Ford-Fulkerson, Sha~~...~~nnon – et. al., ca. 1950).

(a) ~~The~~ We have $\max\{|f| \mid f \text{ is a flow}\}$
$= \min\{c(S,\bar{S}) \mid S \subseteq V, \ s \in S, \ t \notin S\}$.

(b) The same holds if we restrict ourselves to integer flows (i.e., flows f such that $f(a) \in \mathbb{N} \ \forall a \in A$), provided that $c(a) \in \mathbb{N} \ \forall a \in A$.

The proof, which we'll merely outline, rests on the concept of a residual digraph:

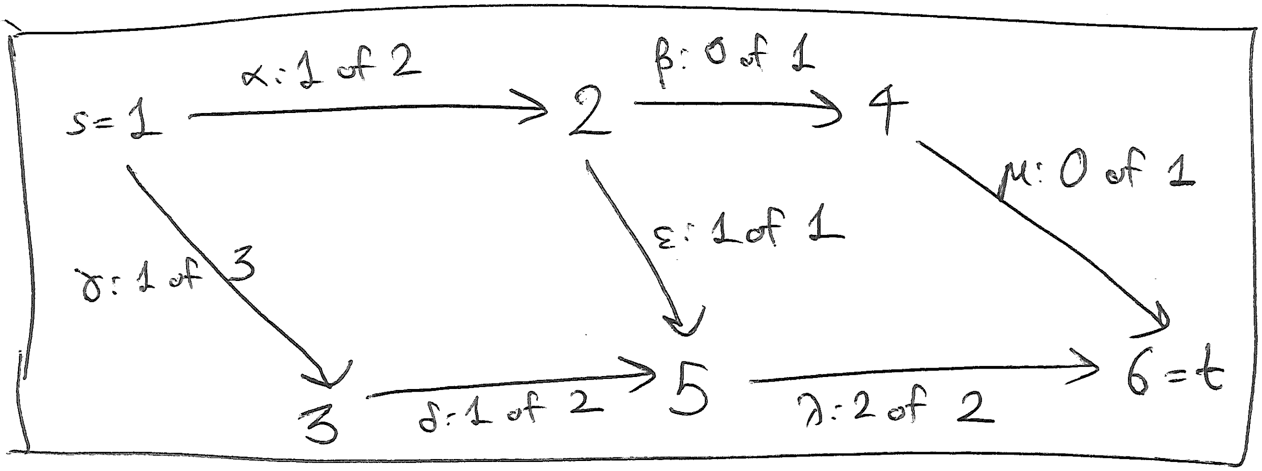
Def. Let $f: A \to \mathbb{Q}_+$ be a flow. The residual digraph D_f

as follows:

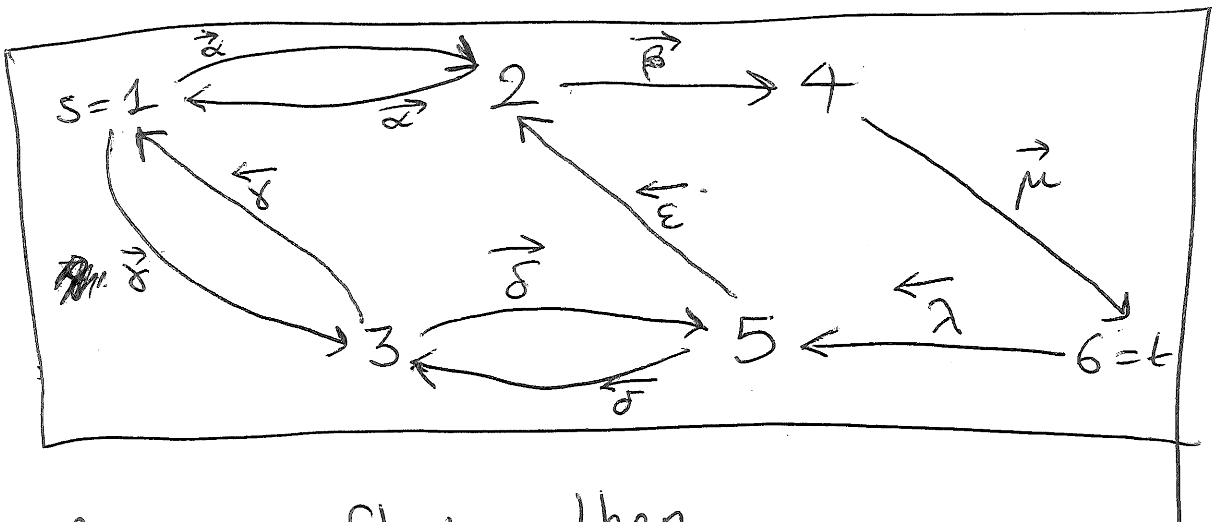
- The vertices of D_f are the $v \in V$.
- The arcs of D_f are:
  - for each arc a of D satisfying $f(a) < c(a)$, we have an arc $\vec{a}$ of D_f having the same source & target as a;
  - for each arc a of D satisfying $f(a) > 0$, we have an arc $\overleftarrow{a}$ of D_f whose source is the target of a & whose target is the source of a.

This digraph D_f represents the changes that can be made to the flow f without breaking the capacity constraints.

Example: Let f be the flow



Then, D_f is



Now, if f is a flow, then

EITHER there is a path $s \to t$ in D_f,

OR there is no such path.

In the former case, ~~the~~ we can obtain a flow f' with $|f'| > |f|$ as follows:

- Start with f.
- ~~For each arc a with~~
- Fix a path p from s to t in D_f.
- For each arc a ~~...~~ with $\vec{a} \in p$, ~~add~~ increase $f(a)$.
- For each arc a with $\overleftarrow{a} \in p$, decrease $f(a)$.

Here, "increase" and "decrease" mean "by the max. possible value" (i.e., by

$$\cancel{\max}\ \min\left(\{f(a) \mid \overleftarrow{a} \in p\} \cup \{c(a) - f(a) \mid \vec{a} \in p\}\right)).$$

In the latter case, let $S = \{v \in S \mid \exists \text{ path } s \to v \text{ in } D_f\}$.
Then, $s \in S$ and $t \notin S$ and $|f| = c(S, \bar{S})$.
Thus, f is a maximum flow by Prop. 7.3 (c).

This gives an algorithm for finding both a maximum flow f and a minimum cut $c(S, \bar{S})$:

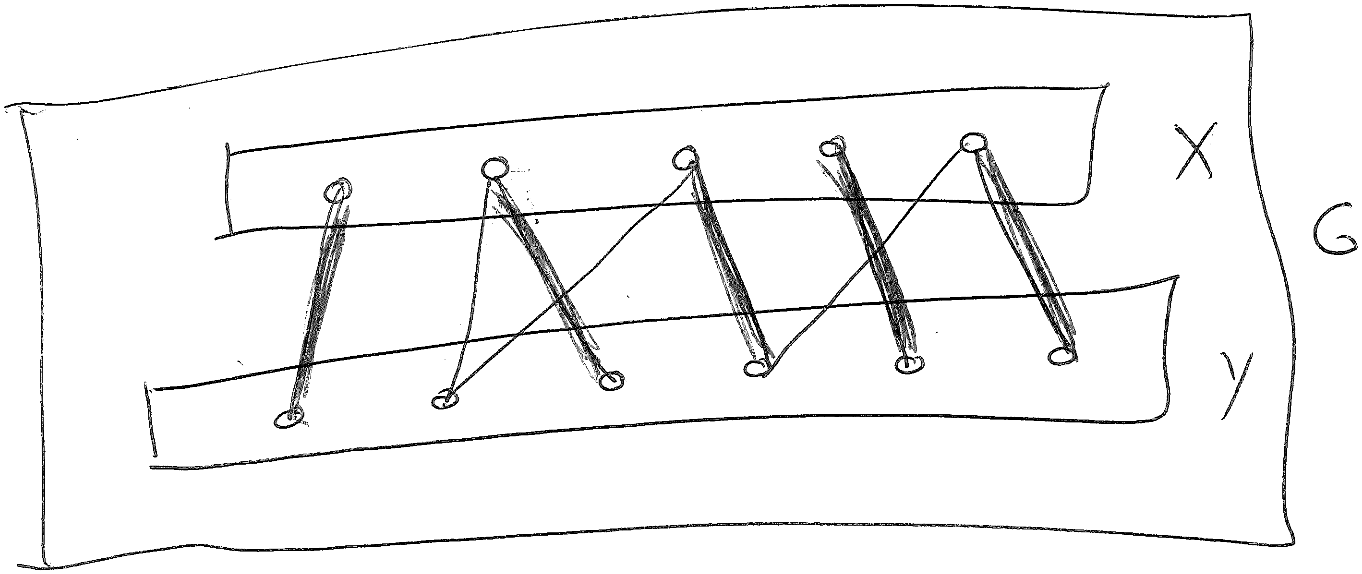
- Start with the "zero flow": $f(a) = 0 \quad \forall a \in A$.
- Do the following:
  - check if $\exists$ path $s \to t$ in D_f;
  - if so, improve f to f', replace ~~by~~ f by f', and repeat.
  - if not, find S as above, and you're done, ($\Rightarrow$ Thm. 7.4 (b)).

This terminates if $c(a) \in \mathbb{N}$ for all $a \in A$. ($\Rightarrow$ Thm. 7.4 (a)).

Also if $c(a) \in \mathbb{Q}$ for all $a \in A$.

Not if $c(a) \in \mathbb{R}$, alas.

Remark: A ~~perfect~~ matching in a bipartite graph $(G; X, Y)$ corresponds to an integer flow in a certain network:



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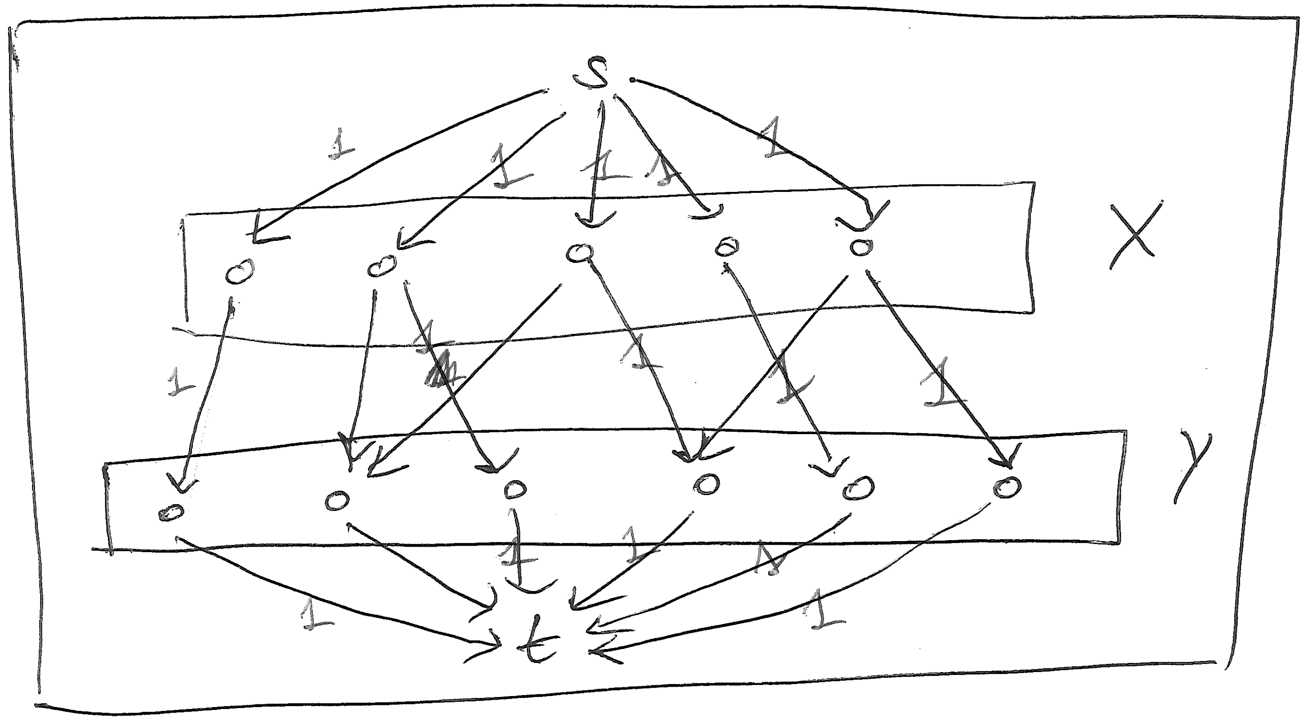

All capacities are 1.