Prop. 5.13. Let $n \in \mathbb{N}$. If a permutation σ has k cycles in its disjoint cycle decomposition (including 1-cycles), then $(-1)^{\sigma} = (-1)^{n-k}$.

Proof. Let notations be as in Thm. 5.12.

Thus $\sigma = (\text{an } n_1\text{-cycle}) \circ (\text{an } n_2\text{-cycle}) \circ \dots \circ (\text{an } n_k\text{-cycle})$,

so
$$(-1)^{\sigma} = (-1)^{(\text{an } n_1\text{-cycle}) \circ (\text{an } n_2\text{-cycle}) \circ \dots \circ (\text{an } n_k\text{-cycle})}$$

$$= (-1)^{\text{an } n_1\text{-cycle}} \cdot (-1)^{\text{an } n_2\text{-cycle}} \cdot \dots \cdot (-1)^{\text{an } n_k\text{-cycle}}$$
(by Thm. 5.10 (d) & induction)

$$= (-1)^{n_1 - 1} \cdot (-1)^{n_2 - 1} \cdot \dots \cdot (-1)^{n_k - 1}$$
(since Thm. 5.10 (c) yields $(-1)^{\text{a } p\text{-cycle}} = (-1)^{p-1}$)

$$= (-1)^{(n_1 - 1) + (n_2 - 1) + \dots + (n_k - 1)} = (-1)^{n-k},$$

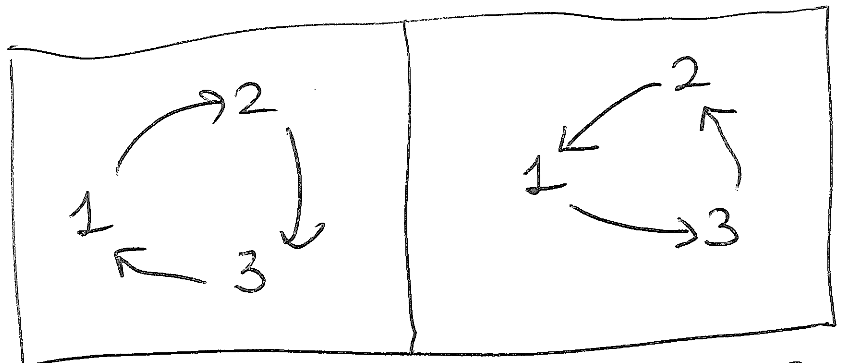
since
$$(n_1 - 1) + (n_2 - 1) + \dots + (n_k - 1) = \underbrace{(n_1 + n_2 + \dots + n_k)}_{=n} - k = n - k.$$
(since $a_{1,1}, a_{1,2}, \dots, a_{k,n_k}$ are just $1, 2, \dots, n$ rearranged) $\square$

Exercise. Let n be a positive integer.

How many n-cycles exist in S_n?

Examples: For $n=2$, the only 2-cycle in S_2 is $\operatorname{cyc}_{1,2} = s_1$.

For $n=3$, the only 3-cycles in S_3 are $\operatorname{cyc}_{1,2,3}$ and $\operatorname{cyc}_{1,3,2}$.



For arbitrary n, the # of n-cycles in S_n is $(n-1)!$.

Idea of proof: There is a bijection

$$\{\text{permutations } \sigma \text{ of } \{2,3,\ldots,n\}\} \to \{n\text{-cycles in } S_n\},$$

$$\sigma \mapsto \operatorname{cyc}_{1,\sigma(2),\sigma(3),\ldots,\sigma(n)}.$$

$\square$

Remark: More generally: Let $n \in \mathbb{N}$. Let $m_1, m_2, \ldots, m_n$ be n nonnegative integers such that $1m_1 + 2m_2 + \cdots + nm_n = n$.

Then, the # of permutations $\sigma \in S_n$ whose disjoint cycle decomposition consists of m_1 1-cycles, m_2 2-cycles, ..., m_n n-cycles is

$$\frac{n!}{m_1!\, m_2! \cdots m_n!\, 1^{m_1}\, 2^{m_2} \cdots n^{m_n}}.$$

(See books on abstract algebra.)

Corollary: $$\sum_{\substack{(m_1, m_2, \ldots, m_n) \in \mathbb{N}^n; \\ 1m_1 + 2m_2 + \ldots + nm_n = n}} \frac{1}{m_1!\, m_2! \cdots m_n!\, 1^{m_1}\, 2^{m_2} \cdots n^{m_n}} = 1.$$

5.5. Determinants

Def. Let $n \in \mathbb{N}$. Let A be an $n \times n$-matrix (e.g. with rational or real or complex entries). Write A as

(1) $$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}.$$

Then, the <u>determinant</u> <u>$\det A$</u> of A is defined by

$$\det A = \sum_{\sigma \in S_n} (-1)^{\sigma} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} .$$

<u>Examples:</u>

$$\det \left(a_{1,1} \right) = a_{1,1} ;$$

$$\det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \underbrace{(-1)^{id}}_{=1} \underbrace{a_{1,id(1)}}_{=a_{1,1}} \underbrace{a_{2,id(2)}}_{=a_{2,2}}$$

$$+ \underbrace{(-1)^{[2,1]}}_{=-1} \underbrace{a_{1,[12](1)}}_{=a_{1,2}} \underbrace{a_{2,[12](2)}}_{=a_{2,1}}$$

$$= a_{1,1} a_{2,2} - a_{1,2} a_{2,1} ;$$

$$\det \begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} = ab'c'' + bc'a'' + ca'b'' - ba'c'' - ac'b'' - cb'a''$$

("Sarrus' rule");

$\det$ (a 4×4-matrix) = (a sum of 12 products) − (another sum of 12 products);

$\det$ (the 0×0-matrix) $= 1$.

<u>Example:</u> I claim that if an $n \times n$-matrix A has two equal rows, then $\det A = 0$.

<u>Proof.</u> Let A have two equal rows. Let row i and row j of A be equal, with $1 \leq i < j \leq n$. Write A as in (1). Then,

$$a_{i,k} = a_{j,k} \qquad \forall k \in [n]. \tag{2}$$

Now, $\det A = \sum_{\sigma \in S_n} (-1)^{\sigma} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$

$$= \sum_{\substack{\sigma \in S_n; \\ \sigma(i) > \sigma(j)}} (-1)^{\sigma} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

$$+ \sum_{\substack{\sigma \in S_n; \\ \sigma(i) < \sigma(j)}} (-1)^{\sigma} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

$$= \sum_{\substack{\tau \in S_n; \\ \tau(i) < \tau(j)}} \underbrace{(-1)^{\tau \circ t_{i,j}}}_{= -(-1)^{\tau}} \underbrace{a_{1,(\tau \circ t_{i,j})(1)} \cdots a_{n,(\tau \circ t_{i,j})(n)}}_{= a_{1,\tau(1)} \cdots \underbrace{a_{i,\tau(j)}}_{\overset{(2)}{=} a_{j,\tau(j)}} \cdots \underbrace{a_{j,\tau(i)}}_{\overset{(2)}{=} a_{i,\tau(i)}} \cdots a_{n,\tau(n)}}$$

$+$ (the second sum)

(here, we substituted $\tau \circ t_{i,j}$ for σ in the first sum)

$$= \sum_{\substack{\tau \in S_n;\\ \tau(i)<\tau(j)}} \left(-(-1)^{\tau}\right) \underbrace{a_{1,\tau(1)} \cdots a_{j,\tau(j)} \cdots a_{i,\tau(i)} \cdots a_{n,\tau(n)}}_{= a_{1,\tau(1)} \cdots a_{n,\tau(n)}}$$

$$+ \sum_{\sigma \in S_n} (-1)^{\sigma} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

$$= \cancel{- \sum_{\substack{\tau \in S_n;\\ \tau(i)<\tau(j)}} (-1)^{\tau} a_{1,\tau(1)} \cdots a_{n,\tau(n)}} + \cancel{\sum_{\sigma \in S_n} (-1)^{\sigma} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}}$$

$$= 0.$$

(See [detnotes, Exercise 5.7] for details & more.) $\square$

<u>Example:</u> Let A be a 5×5-matrix of the form

$$\begin{pmatrix} a & b & c & d & e \\ p & 0 & 0 & 0 & f \\ o & 0 & 0 & 0 & g \\ n & 0 & 0 & 0 & h \\ m & l & k & j & i \end{pmatrix}.$$

Then, $\det A = 0$.

([detnotes, Exercise 5.6 (b)])

Proof. Write A as in (1).

I claim: each $\sigma \in S_5$ satisfies $a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{5,\sigma(5)} = 0$.

Indeed, the 3 numbers $\sigma(2), \sigma(3), \sigma(4)$ must be distinct, and thus cannot all belong to $\{1,5\}$, so one of them must be 2, 3 or 4.

So there exists an $i \in \{2,3,4\}$ with $\sigma(i) \in \{2,3,4\}$.

Then, $a_{i,\sigma(i)} = 0$, so $a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{5,\sigma(5)} = 0$. $\square$

See [detnotes, Ch. 5] for (much) more about determinants.

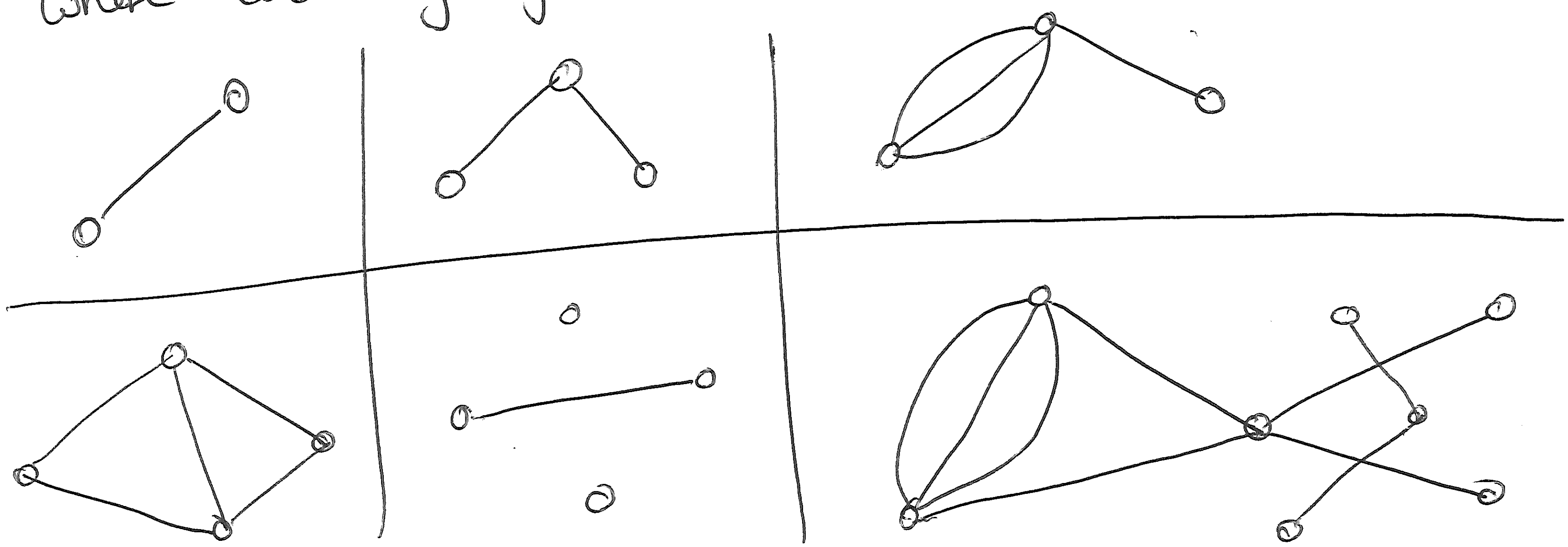
See hw #4 for two exercises in the same spirit.

6. GRAPHS

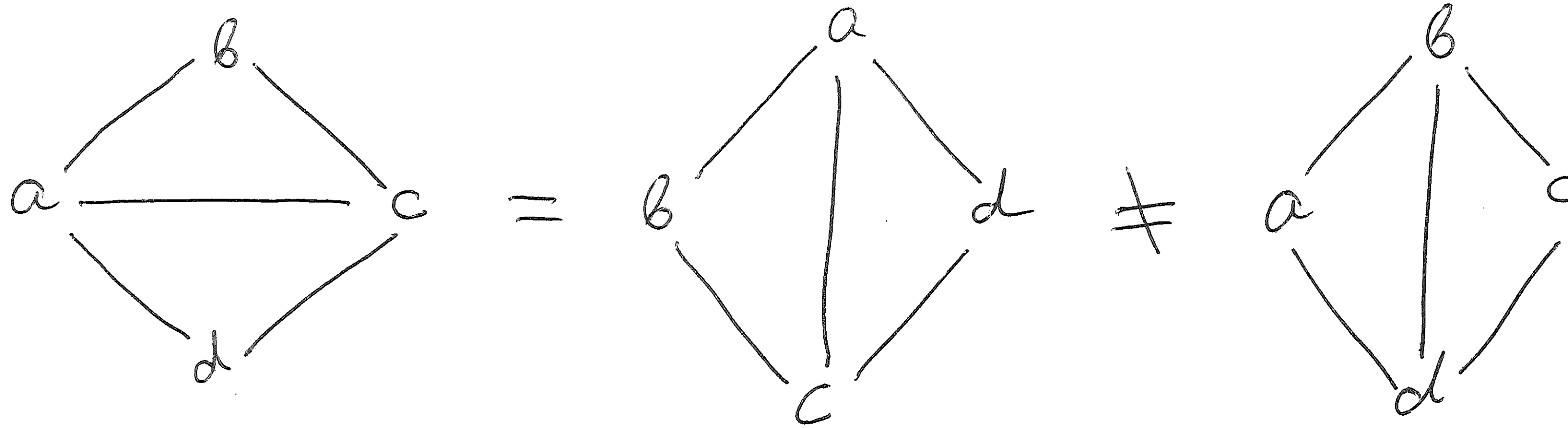
We will only introduce a few notions & prove some basic results. For deeper familiarization with graphs, see [LeLeMe, Ch. 12], [Galvin, §4–§8] (for Cayley's formula), [Guichard], and [nogra] (my notes from Math 5707 in Spring 2017), as well as the books cited in the introduction to [nogra].

6.1. BASICS

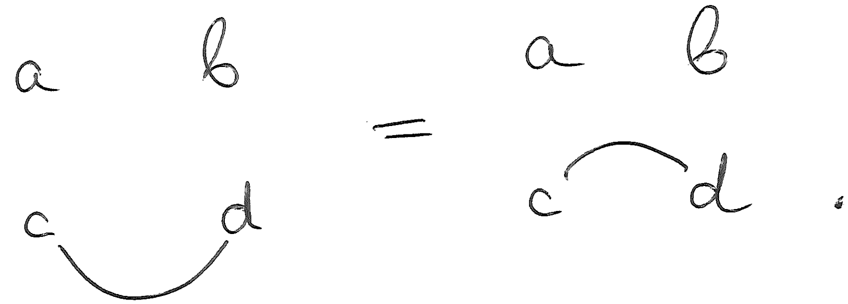
Idea: a graph is a collection of "vertices" and "edges", where each edge joins 2 distinct vertices.



But the vertices are abstract objects, not points in the plane. The pictures above just visually represent graphs. In particular,



and



Rigorous definition:

Def. If S is a set, then $\mathcal{P}_2(S)$ means the set of all 2-element subsets of S.

Def. A *graph* (or, better, *multigraph*) is a triple (V, E, φ), where V and E are finite sets and

$$\varphi : E \to \mathcal{P}_2(V).$$

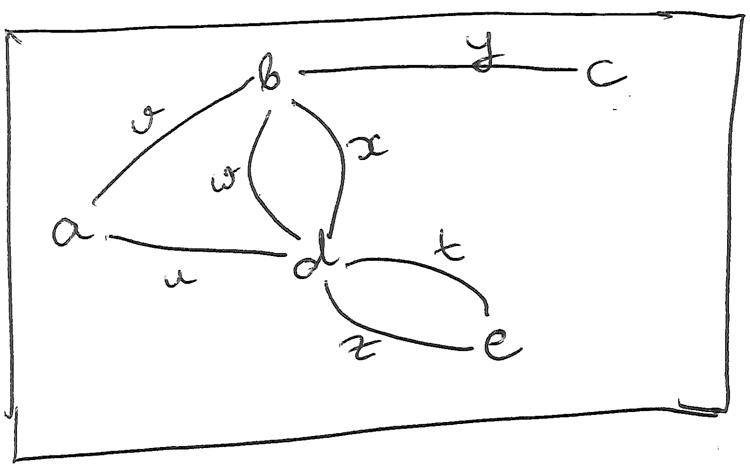
The *vertices* of (V, E, φ) are the elements of V.

The *edges* of (V, E, φ) are the elements of E.

For each edge e, the two elements of $\varphi(e)$ are called the *endpoints* of e, and we say that e *joins* these two elements.

An edge $e \in E$ *contains* (or *passes through*) a vertex v if $v \in \varphi(e)$.

Example:



Here, $V = \{a, b, c, d, e\}$,

$E = \{u, v, w, x, y, z, t\}$,

$\varphi(y) = \{b, c\}$, $\varphi(z) = \{d, e\}$,

$\varphi(t) = \{d, e\}$, ...

To be fully precise, graphs as defined above are called <u>multigraphs</u>. There is also a notion of simple graphs:

<u>Def.</u> A <u>simple graph</u> is a pair (V, E) ~~with~~ where V and E are finite sets such that $E \subseteq \mathcal{P}_2(V)$.

<u>Remark:</u> Multigraphs support "parallel edges" (= multiple edges ~~with~~ joining the same 2 vertices), whereas simple graphs do not.

We can view any simple graph (V, E) as the multigraph $(V, E, \text{"id"})$, where "id": ~~...~~ $E \to \mathcal{P}_2(V)$, $e \mapsto e$.

Some ~~...~~ notions of graph allow loops (= edges with only 1 endpoint); ours don't.

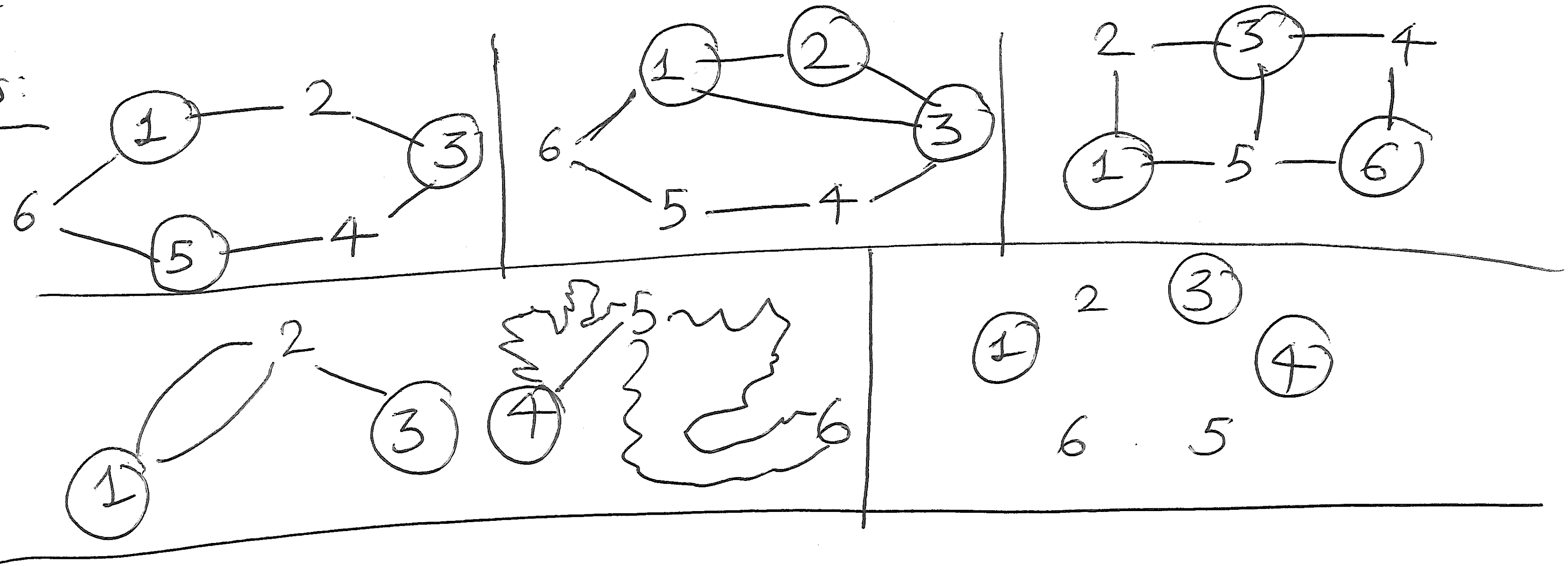
<u>Def.</u> Two vertices u and v of a graph are <u>adjacent</u> if there is an edge with endpoints u and v.

Prop. 6.1. ("$R(3,3) \leq 6$"). Let G be a graph with ≥ 6 vertices. Then,

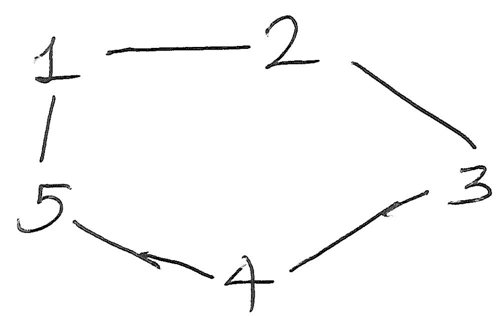
EITHER $\exists$ 3 distinct mutually adjacent vertices in G
OR $\exists$ 3 distinct mutually non-adjacent vertices in G

(or both).

Examples:



Non-example:



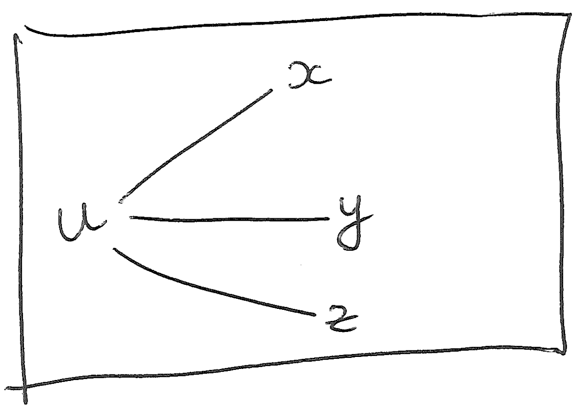
No way!

Def. A *neighbor* of a vertex u of a graph G is any vertex adjacent to u.

Proof of Prop. 6.1. Fix any vertex u of G.

Then, u has either ≥ 3 neighbors or ≥ 3 non-neighbors (not counting u), since G has ≥ 5 vertices distinct from u.

CASE 1: u has ≥ 3 neighbors.

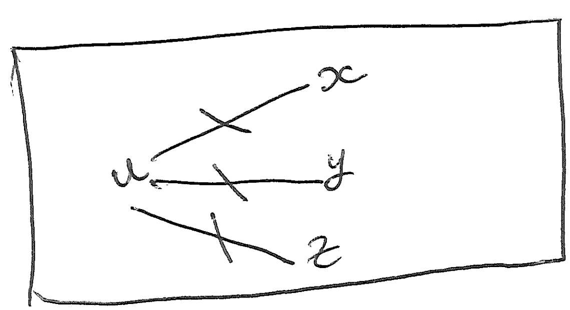


Choose 3 distinct neighbors x, y, z of u.

If two of x, y, z are adjacent to each other, then we get 3 adjacent vertices.

If not, then we get 3 non-adjacent vertices.

CASE 2: u has ≥ 3 non-neighbors.



Same argument as in Case 1 works, except with the roles of "adjacent" and "non-adjacent" interchanged.

$\square$

More generally:

<u>Prop. 6.2. ("Ramsey's theorem for 2 colors).</u> Let $r \in \mathbb{N}$ and $s \in \mathbb{N}$.

Let G be a graph with $\geq \binom{r+s-2}{r-1}$ vertices. Then,

EITHER $\exists\, r$ distinct mutually adjacent vertices in G;

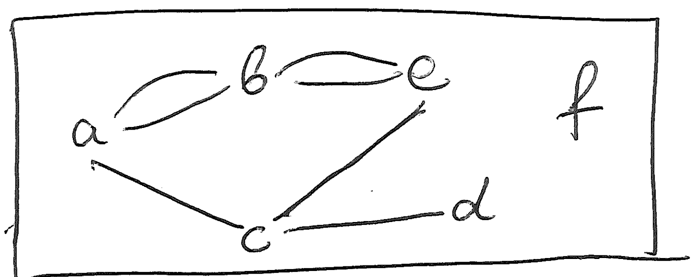
OR $\exists\, s$ distinct mutually non-adjacent vertices in G.

See texts on Ramsey theory and the WP page "Ramsey's theorem". Usually, $\binom{r+s-2}{r-1}$ is not the best bound.

6.2. DEGREES

<u>Def.</u> The <u>degree</u> <u>$\deg v$</u> of a vertex v of a graph is the # of edges containing v.

<u>Example:</u> In the graph



we have

$\deg a = 3$, $\deg b = 4$, $\deg c = 3$, $\deg d = 1$, $\deg e = 3$, $\deg f = 0$.

Prop. 6.3. ("handshaking lemma").

Let $G = (V, E, \varphi)$ be a graph. Then,

$$\sum_{v \in V} \deg v = 2|E|.$$

Proof.

$$\sum_{v \in V} \underbrace{\deg v}_{= \#(\text{edges } e \text{ such that } v \in \varphi(e))}$$

$$= \sum_{v \in V} \#(\text{edges } e \text{ such that } v \in \varphi(e))$$

$$= \#(\text{pairs } (v,e) \text{ such that } v \in \varphi(e))$$

$$= \sum_{e \in E} \underbrace{\#(\text{vertices } v \text{ such that } v \in \varphi(e))}_{=2} = \sum_{e \in E} 2 = 2|E|.$$

$\square$

Prop. 6.4. Let $G = (V, E)$ be a simple graph, with ≥ 2 vertices. Then, G has 2 vertices of equal degree.