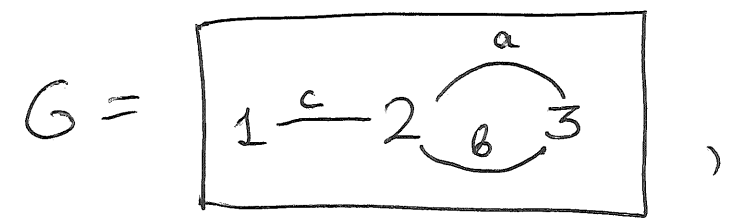


# 6.6. EULERIAN WALKS & CIRCUITS

Def. Let  $G = (V, E, \varphi)$  be a graph.

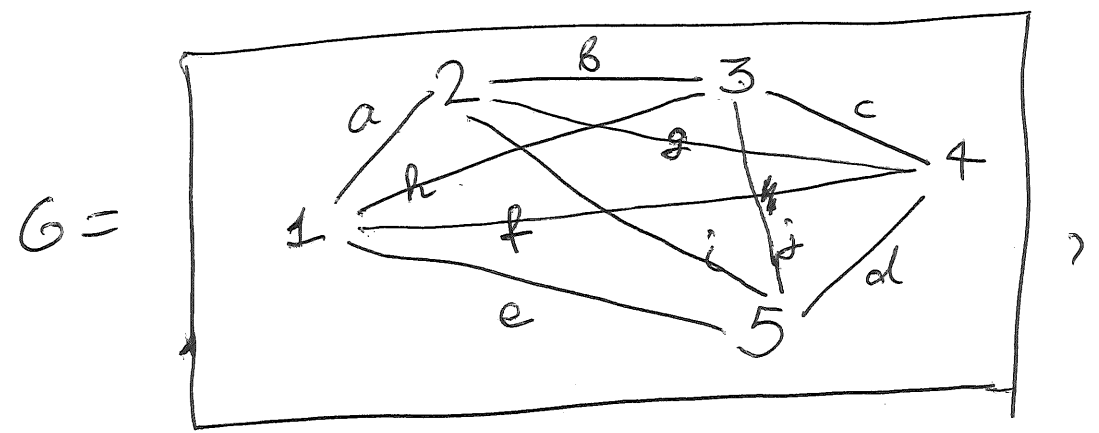
A walk  $(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$  is called Eulerian if each edge of  $G$  appears exactly once among  $e_1, e_2, \dots, e_k$ .

Examples: (a) If



then the walk  $(1, c, 2, a, 3, b, 2)$  is Eulerian.

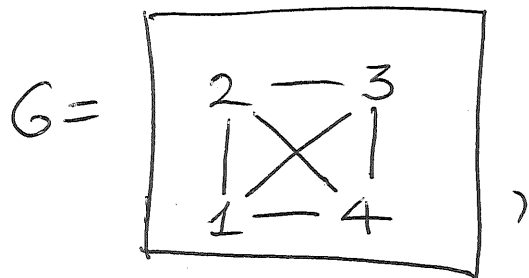
(b) If



then

$(1, a, 2, b, 3, c, 4, d, 5, e, 1, f, 4, g, 2, i, 5, j, 3, h, 1)$  is an Eulerian circuit.

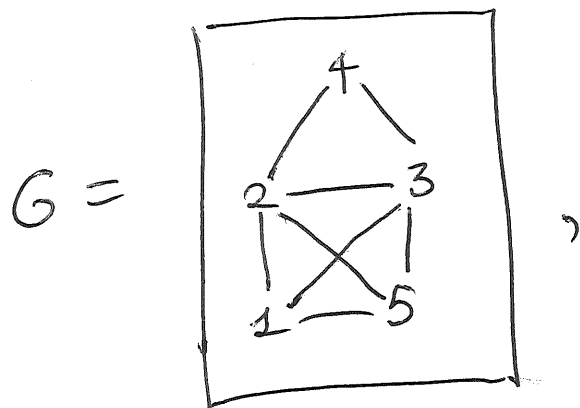
(c) IF



then  $G$  has no Eulerian walks (since it has  $>2$  vertices of odd degree).

-2-

(d) IF

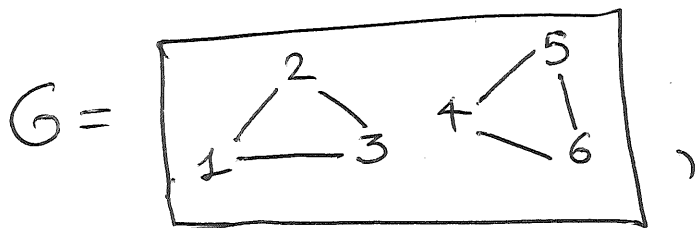


then  $G$  has an Eulerian walk

(e.g.,  $(1, 3, 5, 2, 4, 3, 2, 1, 5)$ , where we omit the edges),

but no Eulerian circuit (since it has 2 vertices of odd degree).

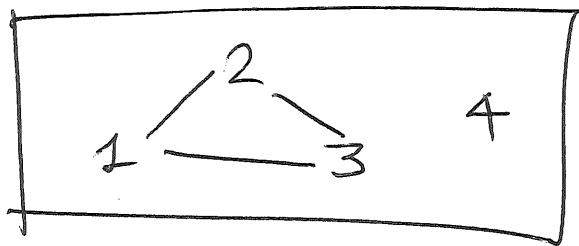
(e) IF



then  $G$  has no Eulerian walk, although all vertices have even degree.

(f)

If  $G =$



, then  $G$  has an Eulerian circuit  $(1, 2, 3, 1)$ , although it is not connected.

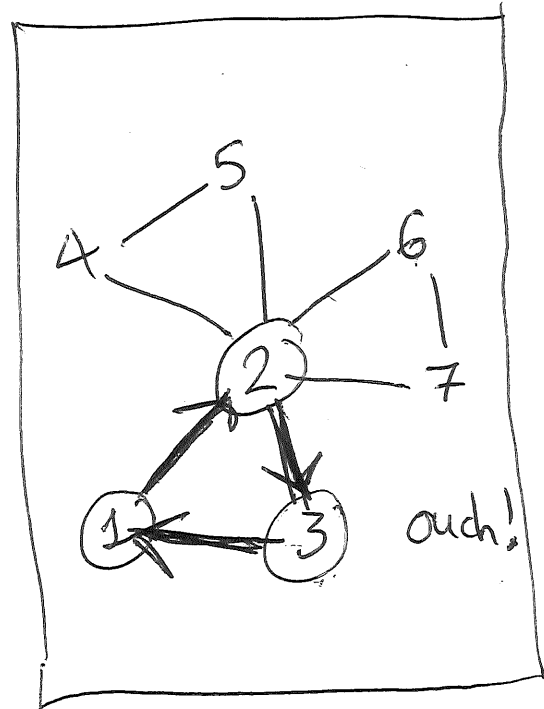
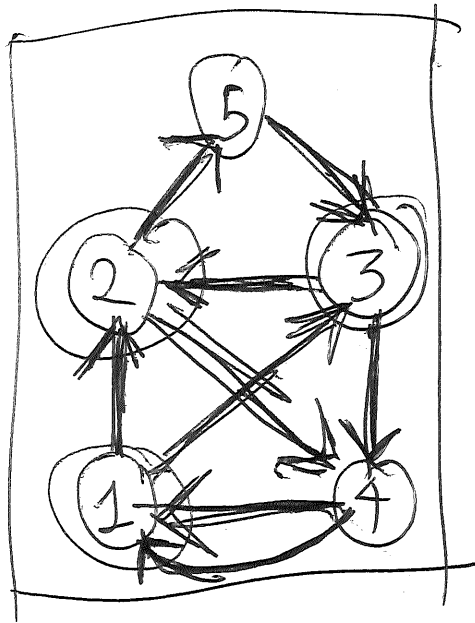
Thm. 6.12. (Euler-Hierholzer), Let  $G$  be a connected graph.

- (a) Then,  $G$  has an Eulerian circuit if & only if each vertex of  $G$  has even degree.
- (b) Then,  $G$  has an Eulerian walk if & only if  $G$  has  $\leq 2$  vertices of odd degree.
- (c) If  $G$  has 2 vertices of odd degree, then these are the starting & the ending point of the walk.

Proof. (2) See [LehMe, Problem 12.44 on p. 563], L-4-

Alternatively, try an algorithm like this (for the "if" part):

- start at some vertex  $v$ ;
- keep travelling along untravelled edges



but this is not enough: we can get stuck (like this).  
~~How do we pick the "right" edges to travel?~~

~~Here is one way:~~

More ~~the~~ complicated reasoning is needed here.

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(E.g., one way to fix this:

If we get stuck, then we get stuck at  $v$   
(for degree reasons)

$\Rightarrow$  we have found a ~~graph~~ circuit  $v \rightarrow v$  that  
contains no edge twice.

Remove the edges of this circuit.

The remaining graph may be disconnected, but  
each of its connected components has an  
Eulerian circuit by induction hypothesis

(induction on  $|E|$ ), since the vertices have  
even degrees. ~~Now~~ Now, "glue" those circuits together.)

(b) follows from (a),

(c) Easy.

□

## 6.7. HAMILTONIAN CYCLES

-6-

Def. Let  $G = (V, E)$  be a simple graph.

A Hamiltonian path of  $G$  means a path of  $G$  that contains each vertex of  $G$  exactly once.

A Hamiltonian cycle of  $G$  means a cycle  $(v_0, v_1, \dots, v_k)$  of  $G$  such that each vertex of  $G$  appears exactly once among  $v_0, v_1, \dots, v_{k-1}$ .  
(omit the edges)

See Math 5707 Spring 2017 Lecture 5 for more about these.

Def. Let  $n \in \mathbb{N}$ . Recall the hypercube graph  $H_n$ ; ~~its~~ its vertices are the elements of  $\{0, 1\}^n$  ("bitstrings of length  $n$ "), and its edges are given by: two bitstrings are adjacent if & only if they differ at only 1 position.

Theorem 6.13. Let  $n \geq 2$ . Then, the graph  $H_n$  has a Hamiltonian ~~exist~~ cycle.

Such circuits are called Gray codes,

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Proof outline, Induction on  $n$ .

Base case:  $n=2$  ✓.

Step:  $n \rightarrow n+1$ ,

Let  $(v_{1,1}, v_{2,1}, \dots, v_{2^n})$  be a Hamiltonian ~~cycle~~ <sup>cycle</sup> for  $H_n$ .

For each ~~each~~  $a = (a_1, a_2, \dots, a_n) \in \{0, 1\}^n$ , we set

$$a^{*}[+1] = (a_1, a_2, \dots, a_n, *1) \text{ and}$$

$$a^{*}[+0] = (a_1, a_2, \dots, a_n, 0).$$

Then,  $(v_{1,1}[+0], v_{2,1}[+0], \dots, v_{2^n,1}[+0], v_{2^n,1}[+1], v_{2^{n-1},1}[+1],$   
 $\dots, v_{1,1}[+1], v_{1,1}[+0])$  is a

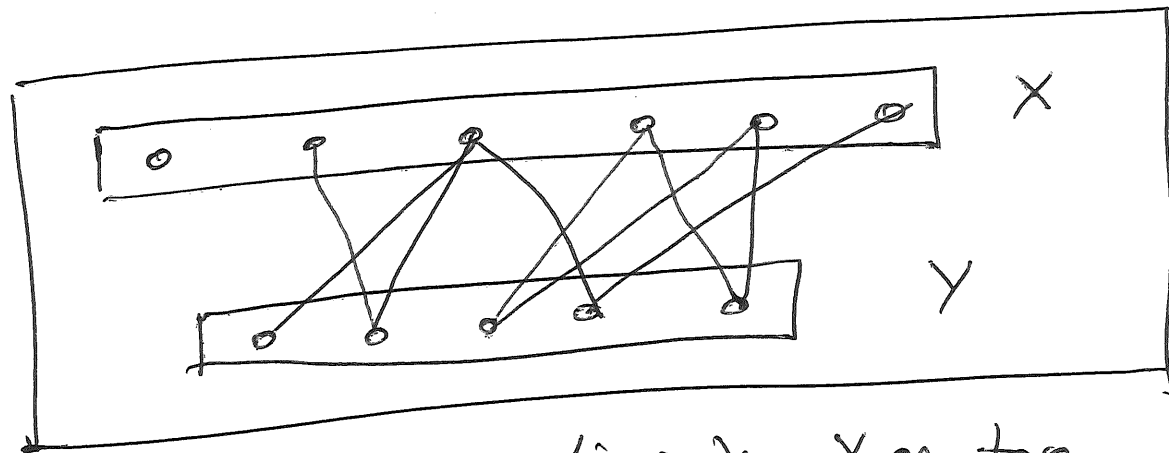
Hamiltonian cycle for  $H_{n+1}$ .

# 6.8. BIPARTITE GRAPHS & HALL'S THEOREM

Def. A bipartite graph is a triple  $(G; X, Y)$ , where  $G = (V, E)$  is a simple graph, and  $X$  and  $Y$  are two subsets of  $V$  such that

- $X \cup Y = V$  and  $X \cap Y = \emptyset$ ;
- each edge  $e \in E$  has exactly one endpoint in  $X$  & exactly one endpoint in  $Y$ .

Example:

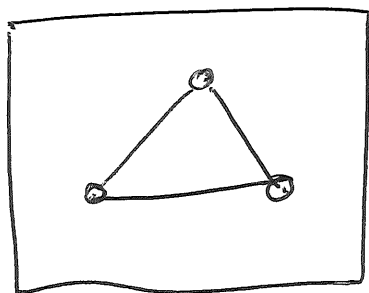


We usually draw the vertices in  $X$  on top  
&  $Y$  on bottom.

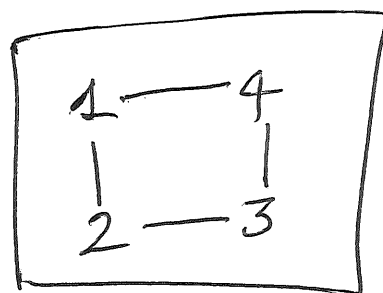
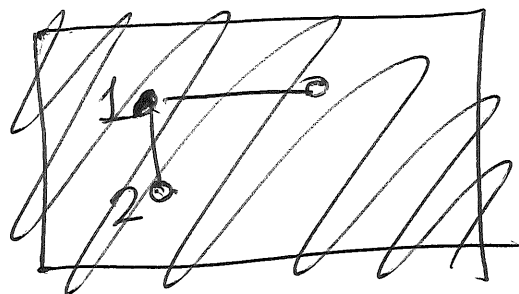


Def. If  $G = (V, E)$  is a simple graph, then a bipartition of  $G$  is a pair  $(X, Y)$  of subsets of  $V$  such that  $(G, X, Y)$  is a bipartite graph. -9-

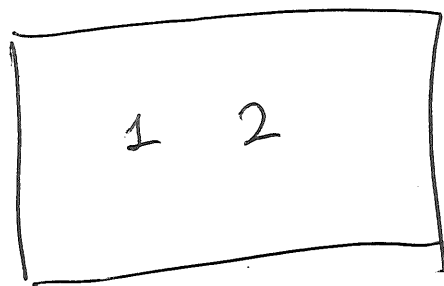
Example:



has no bipartition.



has two bipartitions:  
 $(\{1, 3\}, \{2, 4\})$ ,  
and  $(\{2, 4\}, \{1, 3\})$ .



has four bipartitions:  
 $(\emptyset, \{1, 2\})$ ,  $(\{1\}, \{2\})$ ,  $(\{2\}, \{1\})$ ,  
 $(\{1, 2\}, \emptyset)$ .

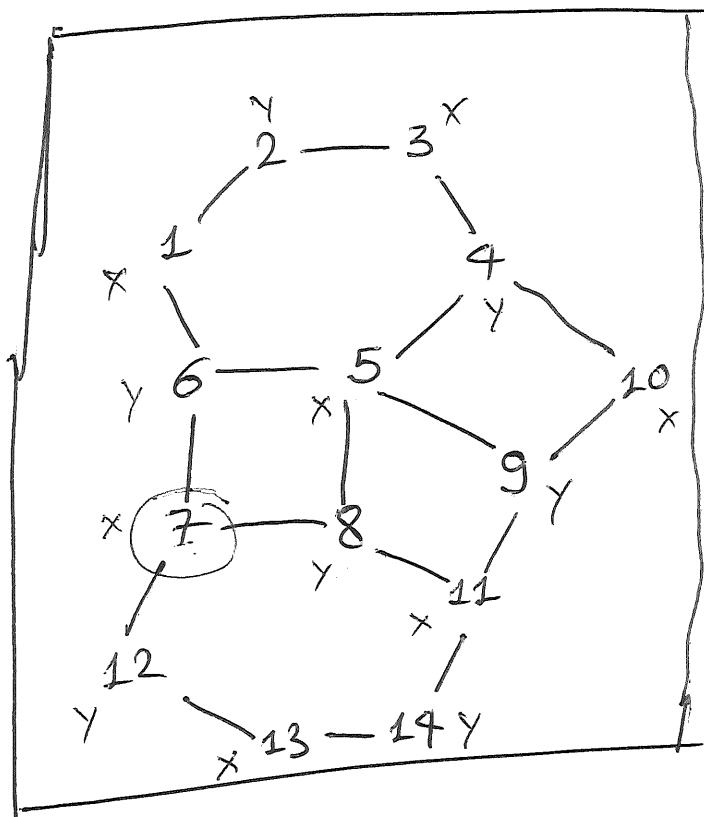
Theorem 6.14. Let  $G = (V, E)$  be a simple graph.

(-10-)

(a) The following are equivalent:

- $G$  has a bipartition.
- All circuits of  $G$  have even length.
- All cycles of  $G$  have even length.

(b) If  $G$  has a bipartition, then the # of bipartitions of  $G$  is  $2^c$ , where  $c$  is the # of connected components of  $G$ .



~~Proof~~

Proof,

See graph theory texts.  $\square$

Def. A matching of a graph  $G$  is a set of disjoint edges of  $G$ . ("Disjoint" means no common vertices.) [-11-]

Examples: (a) If  $G =$  , then

$\{\{1,2\}, \{4,5\}\}$  is a matching of  $G$ ,  
but  $\{\{1,2\}, \{2,3\}\}$  is not.

(b) If  $G =$  , then

$\{\{1,2\}, \{3,4\}\}$  is a matching of  $G$ .

~~(c) If  $n \in \mathbb{N}$ , then the~~

Def. Let  $S$  be a set of vertices of a graph  $G$ .

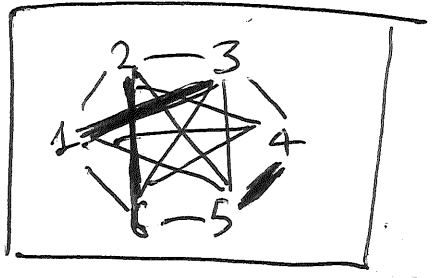
Let  $M$  be a matching of  $G$ .

We say that  $M$  is  $S$ -complete if each ~~the~~ vertex  $v \in S$  is contained in an edge  $e \in M$ .

Def. A matching of a graph  $G=(V, E)$  is perfect if it is  $V$ -complete.

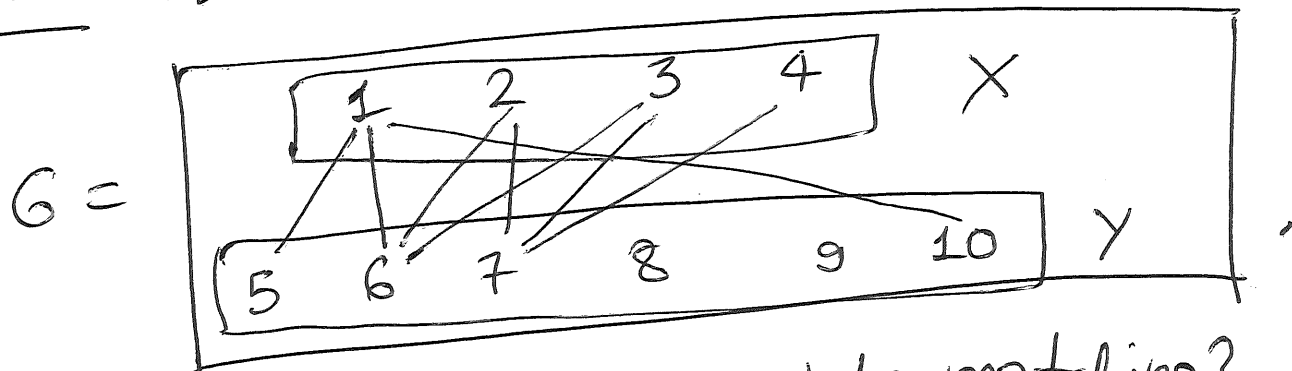
Note that a perfect matching can only exist if  $|V|$  is even.

Remark. Let  $n \in \mathbb{N}$ . ~~A perfect matching of~~ A perfect matching of the complete graph  $K_n$  is the same as a perfect matching of  $[n]$ , as defined in HW #3.



Example: ~~text~~ Consider the bipartite graph

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Does  $G$  have an  $X$ -complete matching?

No, because the 3 vertices 2, 3, 4 have only 2 neighbors altogether.

Thm. 6.15. ("Hall's marriage theorem"). Let  $(G; X, Y)$  be a bipartite graph. Then,  $\exists$   $X$ -complete matching of  $G$  if & only if each subset  $S$  of  $X$  satisfies

$$|N(S)| \geq |S|,$$

where  $N(S) = \{\text{vertices } v \text{ of } G \mid v \text{ has 2 neighbor in } S\}$ .

For an elementary proof, see [DeleMe, Thm. 12.5.2].