University of Minnesota, School of Mathematics

Math 5705: Enumerative Combinatorics, Fall 2018: Homework 1

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due date: **Wednesday, 19 September 2018** at the beginning of class, or before that by email or canvas.
Please solve **at most 5 of the 7 exercises**!

1 Exercise 1

1.1 Problem

Let n be a positive integer.

An n-tuple $(i_1, i_2, \ldots, i_n) \in \{0, 1, 2, 3\}^n$ is said to be *even* if the sum $i_1 + i_2 + \cdots + i_n$ is even. (For example, the 4-tuple $(2, 3, 1, 2)$ is even, whereas $(1, 2, 3, 1)$ is not.)

Compute the number of all even n-tuples $(i_1, i_2, \ldots, i_n) \in \{0, 1, 2, 3\}^n$.

(Here and in all future exercises, all answers need to be proven.)

[**Hint:** Compare with Exercise 3 on Homework set #0.]

1.2 Solution

[...]

2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$.

An n-tuple $(i_1, i_2, \ldots, i_n) \in \{0, 1, 2\}^n$ is said to be *even* if the sum $i_1 + i_2 + \cdots + i_n$ is even. (For example, the 4-tuple $(2, 1, 1, 2)$ is even, whereas $(1, 2, 2, 2)$ is not.)

Let e_n be the number of all even n-tuples $(i_1, i_2, \ldots, i_n) \in \{0, 1, 2\}^n$.

Prove that $e_n = \dfrac{3^n + 1}{2}$.

[**Hint:** Induction on n.]

2.2 Solution

[...]

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3 Exercise 3

3.1 Problem

For any real number x and any $k \in \mathbb{N}$, we define the lower factorial $x^{\underline{k}}$ as in Exercise 2 of Homework set #0. (Thus, $x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1) = \prod_{i=0}^{k-1}(x-i)$. This boils down to $x^{\underline{0}} = 1$ when $k = 0$, since empty products are defined to be 1.)

Let k, a and b be three positive integers such that $k \leq a \leq b$. Prove that

$$(k-1)\sum_{i=a}^{b}\frac{1}{i^{\underline{k}}} = \frac{1}{(a-1)^{\underline{k-1}}} - \frac{1}{b^{\underline{k-1}}}. \tag{1}$$

3.2 Remark

Remark 3.1. This is similar to Exercise 2 of Homework set #0, but here the lower factorials are in the denominators. The analogous fact from calculus is

$$(k-1)\int_a^b \frac{1}{x^k}dx = \frac{1}{a^{k-1}} - \frac{1}{b^{k-1}}.$$

3.3 Solution

[...]

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4 Exercise 4

4.1 Problem

Definition 4.1. The *Fibonacci sequence* is the sequence $(f_0, f_1, f_2, \ldots)$ of integers which is defined recursively by $f_0 = 0$, $f_1 = 1$, and

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for all } n \geq 2. \tag{2}$$

Here is a table of some of its first terms:

n	0	1	2	3	4	5	6	7	8	9
f_n	0	1	1	2	3	5	8	13	21	34

Let $n \in \mathbb{N}$. Recall some definitions from class:

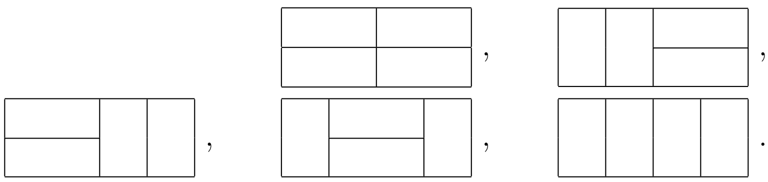
Let $R_{n,2}$ denote the set $[n] \times [2]$, which we regard as a rectangle of width n and height 2 (by identifying the squares with pairs of coordinates).

A *vertical domino* is a set of the form $\{(i, j), (i, j+1)\}$ for some $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$.

A *horizontal domino* is a set of the form $\{(i, j), (i+1, j)\}$ for some $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$.

A *domino tiling* of $R_{n,2}$ means a set of disjoint dominos (i.e., vertical dominos and horizontal dominos) whose union is $R_{n,2}$.

For example, there are 5 domino tilings of $R_{4,2}$, namely



Written as a set of dominos, the second of these tilings is

$$\{\{(1,1), (1,2)\}, \{(2,1), (2,2)\}, \{(3,1), (4,1)\}, \{(3,2), (4,2)\}\}.$$

We have seen in class (September 5) that

$$\text{the number of domino tilings of } R_{n,2} \text{ is } f_{n+1}. \tag{3}$$

We have also counted "axisymmetric" domino tilings.

Let us now define a different kind of symmetry: A domino tiling S of $R_{n,2}$ is said to be *centrosymmetric* if reflecting it across the center of the rectangle $R_{n,2}$ leaves it unchanged. (Formally, if S is regarded as a set, it means that for every domino $\{(i, j), (i', j')\} \in S$, its "opposite domino" $\{(n+1-i, 3-j), (n+1-i', 3-j')\}$ is also in S.) For example, among the 5 domino tilings of $R_{4,2}$ listed above, exactly 3 are centrosymmetric (namely, the first, the fourth and the fifth).

Let s_n be the number of centrosymmetric domino tilings of $R_{n,2}$.

(a) Prove that $s_n = f_{(n+1)/2}$ if n is odd.

(b) Prove that $s_n = f_{n/2+2}$ if n is even.

(Note that these are the same numbers as for axisymmetric domino tilings!)
[**Hint:** This is a bit of a trick problem.]

4.2 Solution

[...]

5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. Let $S_{n,2}$ be the set

$$([n+1] \times [2]) \setminus \{(1,2), (n+1,1)\}.$$

For example, here is how $S_{6,2}$ looks like:

Find the number of domino tilings of $S_{n,2}$.

5.2 Solution

[...]

6 Exercise 6

6.1 Problem

Let $n \in \mathbb{N}$. If S is a finite nonempty set of integers, then $\max S$ denotes the maximum of S (that is, the largest element of S).

(a) Find the number of nonempty subsets S of $[n]$ satisfying $\max S = |S|$.

(b) Find the number of nonempty subsets S of $[n]$ satisfying $\max S = |S| + 1$.

[**Hint:** Exercise 7 from Spring 2018 Math 4707 Homework set #1 is similar to part **(a)**, but uses the minimum instead of the maximum. Does this mean the answer should be similar?]

6.2 Solution

[...]

7 Exercise 7

7.1 Problem

For any nonnegative integers a and b and any real x, prove that

$$x^{\underline{a}}x^{\underline{b}} = \sum_{r=\max\{a,b\}}^{a+b} \frac{a!b!}{(r-a)!\,(r-b)!\,(a+b-r)!} x^{\underline{r}}. \tag{4}$$

[**Hint:** Induction. First show the identities $x^{\underline{k}} = x^{\underline{k-1}}(x-k+1)$ and $xx^{\underline{k}} = x^{\underline{k+1}} + kx^{\underline{k}}$.]

7.2 Solution

[...]