Petersen & Tenner, "The depth of a permutation"
for HW5 exercise 6.

Continuing the proof of Thm. 8.6 (the "$\square$" was premature):

$\Longleftarrow$: Assume $[x^0]a$ has an inverse in K.
Write a as $a = (a_0, a_1, a_2, \ldots)$, and try to find an FPS $b = (b_0, b_1, b_2, \ldots)$ with $ab = 1$.

So we want

$$(1, 0, 0, 0, \ldots) = 1 = ab = (a_0, a_1, a_2, \ldots)(b_0, b_1, b_2, \ldots)$$
$$= (a_0 b_0, \ a_0 b_1 + a_1 b_0, \ a_0 b_2 + a_1 b_1 + a_2 b_0, \ldots).$$

So we want

$$\begin{cases} 1 = a_0 \underline{b_0}, \\ 0 = a_0 \underline{b_1} + a_1 b_0, \\ 0 = a_0 \underline{b_2} + a_1 b_1 + a_2 b_0, \\ 0 = a_0 \underline{b_3} + a_1 b_2 + a_2 b_1 + a_3 b_0, \\ \ldots\ldots \end{cases}$$

Solve this system by elimination: get b_0 from 1st eqn, then b_1 from the next, etc.

This can be done, since $a_0 = [x^0]a$ has a mult. inverse and thus can be divided by. $\square$

<u>Def.</u> If $a \in K[[x]]$ has a (mult.) inverse a^{-1}, then we can define a^{-n} for each $n \in \mathbb{N}$ by setting $a^{-n} = (a^{-1})^n$.

So the powers a^k make sense $\forall k \in \mathbb{Z}$.

<u>Thm. 8.7.</u> The FPS $1-x$ has an inverse, which is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots .$$

<u>1st proof.</u> $(1-x)(1+x+x^2+x^3+\dots)$

$$= (1 + \cancel{x} + \cancel{x^2} + \cancel{x^3} + \dots) - (\cancel{x} + \cancel{x^2} + \cancel{x^3} + \dots) = 1. \quad \square$$

2nd proof. $(1-x)(1+x+x^2+x^3+\cdots)$

$$= (1-\cancel{x}) + (\cancel{x}-\cancel{x^2}) + (\cancel{x^2}-\cancel{x^3}) + (\cancel{x^3}-\cancel{x^4}) + \cdots = 1. \qquad \square$$

Thm. 8.8 ("Newton's binomial theorem").

$$(1+x)^n = \underbrace{\sum_k \binom{n}{k} x^k}_{\text{actually an } \infty \text{ sum if } n<0} \qquad \forall n \in \mathbb{Z},$$

Proof idea: For $n \geq 0$, this follows from the regular binomial thm.
For $n=-1$, this says $(1+x)^{-1} = \sum_k (-1)^k x^k$, which is similar to Thm. 8.7.

Lem. 8.9. $$(1+x)^{-n} = \sum_k (-1)^k \binom{n+k-1}{k} x^k \qquad \forall n \in \mathbb{N}.$$

Proof idea for Lem. 8.9. Induction on n.
For the ind. step, we need to check:

$$\left(\sum_k (-1)^k \binom{n+k-1}{k} x^k\right) \cdot (1+x)^{-1} = \sum_k (-1)^k \binom{n+k}{k} x^k .$$

Equivalently,

$$\sum_k (-1)^k \binom{n+k-1}{k} x^k = \left(\sum_k (-1)^k \binom{n+k}{k} x^k\right) \cdot (1+x) .$$

The RHS ~~becomes~~ can be rewritten as

$$\sum_k (-1)^k \binom{n+k}{k} x^k + \underbrace{\sum_k (-1)^k \binom{n+k}{k} x^{k+1}}_{= \sum_k (-1)^{k-1} \binom{n+k-1}{k-1} x^k}$$

$$= \sum_k \underbrace{\left[(-1)^k \binom{n+k}{k} + (-1)^{k-1} \binom{n+k-1}{k-1}\right]}_{\begin{aligned} &= (-1)^k \left(\binom{n+k}{k} - \binom{n+k-1}{k-1}\right) \\ &= (-1)^k \binom{n+k-1}{k} \quad \text{(by the recurrence of binomial coefficients)} \end{aligned}} x^k$$

$$= \sum_k (-1)^k \binom{n+k-1}{k} x^k,$$

which is the LHS. $\square$

8.5. Substitution

~~Prop.~~ **Def.** Let f and g be two FPS with $[x^0] g = 0$ (that is, $g = g_1 x + g_2 x^2 + g_3 x^3 + \ldots$).

Then, the FPS $f \circ g$ (also known as $f[g]$) is defined as follows:

Write f as $f = \sum_{n \geq 0} f_n x^n$, and set $f \circ g = \sum_{n \geq 0} f_n g^n$.

We call $f \circ g$ the <u>composition</u> of f with g, or the result of <u>substituting</u> g for x in f.

We will NOT call it $f(g)$, to avoid clashing with product notation.

Rmk. The sum $\sum_{n\geq 0} f_n g^n$ in the above definition is well-defined, i.e., the family $(f_n g^n)_{n\in\mathbb{N}}$ is summable, since

(86) the first n coefficients of g^n are 0 $\forall n \in \mathbb{N}$.

((86) is easy to prove by induction on n.
Alternatively: Write $g = xh$ for some FPS h, since $[x^0]g = 0$. Thus, $g^n = x^n h^n$.)

Example: We can substitute $x+x^2$ for x into $1+x+x^2+\dots$.

The result is

$$1+(x+x^2)+(x+x^2)^2+(x+x^2)^3+(x+x^2)^4+\dots$$

$$= 1+x+2x^2+3x^3+5x^4+8x^5+\dots$$

$$= \sum_{n\geq 0} f_{n+1} x^n, \quad \text{where } f_i = \text{Fibonacci \#s}.$$

This is because substituting $x+x^2$ for x in

$$1 + x + x^2 + \cdots = \frac{1}{1-x}$$

yields $\quad 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \cdots$

$$= \frac{1}{1-(x+x^2)} = \frac{1}{1-x-x^2} \underset{§8.1}{\overset{\text{Ex. 1}}{=\!=}} \sum_{n\geq 0} f_{n+1} x^n .$$

Here, we have tacitly used:

Prop. 8.10. Substitution satisfies the rules you would expect:

- $(g_1+g_2)\circ h = g_1 \circ h + g_2 \circ h$
- $(g_1 g_2) \circ h = (g_1 \circ h)(g_2 \circ h)$
- $f \circ (g \circ h) = (f \circ g) \circ h$

$\Bigg\}$ when everything is well-defined

(See [Loehr, Ch. 7] for details.)

This all justifies Ex. 1 in §8.1.

Rmk. A polynomial is a FPS $(a_0, a_1, a_2, \ldots)$ such that all but finitely many $i \in \mathbb{N}$ satisfy $a_i = 0$.

To justify Ex. 2, we need to define $(1+x)^n$ for $n \notin \mathbb{Z}$.

Option 1: define $(1+x)^n$ as $\sum_k \binom{n}{k} x^k$.

But then, we would have to prove all the rules of exponents: $(1+x)^n (1+x)^m = (1+x)^{n+m}$,

$((1+x)^n)^m = (1+x)^{nm}$,

etc.

Option 2: define $(1+x)^n$ as $\exp(n \log(1+x))$.

What are exp & log? We define

$$\exp = \sum_{n \geq 0} \frac{1}{n!} x^n, \qquad \log(1+x) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} x^n.$$

So $(1+x)^n = \exp \circ (n \cdot \log(1+x))$.

You would still have to prove many things, but this is more doable. See [Loehr] or [my log/exp notes from Fall 2017 Math 4990].

(Note: You need K to contain the rational numbers.)

8.6. Another example

Here is another application of FPS; see [Galvin, §40 (1)].

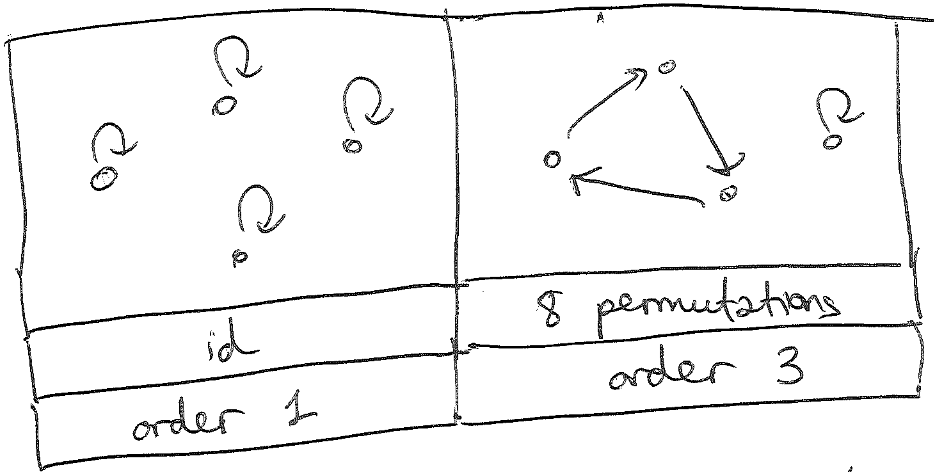
Def. Let $n \in \mathbb{N}$, Let $\sigma \in S_n$. The *order* of σ is the smallest $k > 0$ such that $\sigma^k = \mathrm{id}$.

Prop. 8.11. Let $n \in \mathbb{N}$ and $\sigma \in S_n$. The order of σ is well-defined & equals the lcm (= least common multiple) of the lengths of the cycles of σ.

Proof idea. You just need to show that an $m \in \mathbb{N}$ satisfies $\sigma^m = \mathrm{id}$ if & only if the length of each cycle of σ divides m. $\square$

~~Let~~ For each $n \in \mathbb{N}$, let a_n be the # of permutations $\sigma \in S_n$ having odd order. What is a_n?

Ex: $n = 4$:



$\Longrightarrow a_4 = 9$.

Similarly, $a_3 = 3$ and $a_2 = 1$ and $a_5 = 45$.
(OEIS: A000246.)

First observation: A perm. $\sigma \in S_n$ has odd order $\Longleftrightarrow$ all cycles of σ have odd lengths. Thus,

$$a_n = \sum_{\substack{(i_1, i_2, \ldots, i_n) \in \mathbb{N}^n; \\ i_k = 0 \ \forall \text{ even } k; \\ 1i_1 + 2i_2 + \cdots + ni_n = n}} \underbrace{(\# \text{ of } \sigma \in S_n \text{ with } i_j \text{ cycles of length } j)}_{= \frac{n!}{i_1! i_2! \cdots i_n! \, 1^{i_1} 2^{i_2} \cdots n^{i_n}}}$$

(by Exercise after Prop. 4.13)

$$= \sum_{\substack{(i_1, i_2, \ldots, i_n) \in \mathbb{N}^n; \\ i_k = 0 \ \forall \text{ even } k; \\ 1 i_1 + 2 i_2 + \cdots + n i_n = n}} \frac{n!}{i_1! i_2! \cdots i_n! 1^{i_1} 2^{i_2} \cdots n^{i_n}}$$

~~Divide this Multiply this by~~ $\cancel{\frac{x^n}{n!}}$

$$= \sum_{\substack{(i_1, i_2, i_3, \ldots) \in \mathbb{N}^\infty; \\ i_k = 0 \ \forall \text{ even } k; \\ 1 i_1 + 2 i_2 + \cdots = n}} \frac{n!}{i_1! i_2! \cdots 1^{i_1} 2^{i_2} \cdots}$$

(since $1 i_1 + 2 i_2 + \cdots = n$ forces $i_{n+1}, i_{n+2}, i_{n+3}, \ldots$ to be 0, so we don't really get any addends).

Multiply this ~~ugly~~ identity by $\frac{x^n}{n!}$ to get

$$x^n = x^{1i_1+2i_2+\dots} = x^{1i_1}x^{2i_2}\cdots$$

$$\frac{a_n x^n}{n!} = \sum_{\substack{(i_1,i_2,i_3,\dots)\in\mathbb{N}^\infty;\\ i_k=0\ \forall\text{ even } k;\\ 1i_1+2i_2+\dots=n}} \frac{x^n}{i_1!\,i_2!\cdots 1^{i_1}2^{i_2}\cdots}$$

$$= \sum_{\substack{(i_1,i_2,i_3,\dots)\in\mathbb{N}^\infty;\\ i_k=0\ \forall\text{ even } k;\\ 1i_1+2i_2+\dots=n}} \frac{x^{1i_1}x^{2i_2}\cdots}{i_1!\,i_2!\cdots 1^{i_1}2^{i_2}\cdots}.$$

Summing this over all $n\in\mathbb{N}$, we get

$$\sum_{n\in\mathbb{N}} \frac{a_n x^n}{n!} = \sum_{\substack{(i_1,i_2,i_3,\dots)\in\mathbb{N}^\infty;\\ i_k=0\ \forall\text{ even } k;\\ i_k=0 \text{ for all but}\\ \text{finitely many } k}} \frac{x^{1i_1}x^{2i_2}\cdots}{i_1!\,i_2!\cdots 1^{i_1}2^{i_2}\cdots}$$

$$= \sum_{\substack{(i_1, i_3, i_5, \ldots) \in \mathbb{N}^\infty; \\ i_k = 0 \text{ for all but} \\ \text{finitely many } k}} \frac{x^{1i_1} x^{3i_3} \cdots}{i_1! i_3! \cdots 1^{i_1} 3^{i_3} \cdots}$$

$$= \left(\sum_{i_1 \in \mathbb{N}} \frac{x^{1i_1}}{i_1! 1^{i_1}} \right) \left(\sum_{i_3 \in \mathbb{N}} \frac{x^{3i_3}}{i_3! 3^{i_3}} \right) \cdots\cdots$$

(note that infinite products of FPSs can make sense just as infinite sums do)

$$= \prod_{k \geq 1 \text{ odd}} \underbrace{\sum_{i \in \mathbb{N}} \frac{x^{ki}}{i! k^i}}_{= \sum_{i \in \mathbb{N}} \frac{(x^k/k)^i}{i!} = \exp \circ (x^k/k)} = \prod_{\substack{k \geq 1 \\ \text{odd}}} \left(\exp \circ (x^k/k) \right)$$

$$= \exp \circ \left(\sum_{\substack{k \geq 1 \\ \text{odd}}} x^k / k \right)$$

(here we used the rule $\cancel{\prod_{k\geq 1} \exp}$

$$\prod_{i \in I} \exp \circ f_i = \exp \circ \left(\sum_{i \in I} f_i \right),$$

which is not hard to check).

"destructive interference"

But

$$\sum_{\substack{k \geq 1 \\ \text{odd}}} x^k / k = \frac{1}{2} \left(\underbrace{\sum_{k \geq 1} x^k / k}_{= -\log(1-x)} - \underbrace{\sum_{k \geq 1} (-x)^k / k}_{= -\log(1+x)} \right)$$

$$= \frac{1}{2} \left(-\log(1-x) - \left(-\log(1+x) \right) \right)$$

$$= \frac{1}{2} \left(\log(1+x) - \log(1-x) \right),$$

so this becomes

$$\sum_{n\in\mathbb{N}} \frac{a_n x^n}{n!} = \exp \circ \left(\frac{1}{2} \left(\log(1+x) - \log(1-x) \right) \right)$$

$$= \left(\frac{1+x}{1-x} \right)^{1/2}$$

$$= \frac{\left((1+x)(1-x)\right)^{1/2}}{1-x} = \frac{(1-x^2)^{1/2}}{1-x}$$

$$= \underbrace{(1-x^2)^{1/2}}_{=\sum_{k\geq 0} \binom{1/2}{k} (-x^2)^k} \cdot \underbrace{(1-x)^{-1}}_{=\sum_{k\geq 0} x^k}$$

$$= \left(\sum_{k\geq 0} \binom{1/2}{k} (-x^2)^k \right) \left(\sum_{k\geq 0} x^k \right)$$

$$= \left(\sum_{k\geq 0} \binom{1/2}{k} (-1)^k x^{2k} \right) \left(\sum_{k'\geq 0} x^{k'} \right)$$

$$= \sum_{n \geq 0} \left(\sum_{k \leq n/2} \binom{1/2}{k} (-1)^k \right) x^n .$$

Comparing coefficients, we get

$$\frac{a_n}{n!} = \sum_{k \leq n/2} \binom{1/2}{k} (-1)^k = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{1/2}{k}$$

$$= (-1)^{\lfloor n/2 \rfloor} \binom{1/2 - 1}{\lfloor n/2 \rfloor}$$

(by HW #2 exercise 4, applied to $\lfloor n/2 \rfloor$ and $1/2$ instead of m and n (since we did not need $n \in \mathbb{N}$ in that exercise))

$$= (-1)^{\lfloor n/2 \rfloor} \binom{-1/2}{\lfloor n/2 \rfloor} \underset{\text{exe } 3(a)}{\overset{\text{HW\#3}}{=}} \underbrace{(-1)^{\lfloor n/2 \rfloor} \left(\frac{-1}{4}\right)^{\lfloor n/2 \rfloor}}_{= (1/4)^{\lfloor n/2 \rfloor}} \binom{2\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor}$$

$$= (1/4)^{\lfloor n/2 \rfloor} \binom{2\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor}.$$

Thus,

$$a_n = n! \cdot (1/4)^{\lfloor n/2 \rfloor} \binom{2\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor}.$$

9. Partitions

9.1. Basics:

Recall (from §3.6): A <u>partition</u> of an $n \in \mathbb{Z}$ means a weakly decreasing tuple of pos. ints. with sum n.

The entries of a partition are called its <u>parts</u>.

$p(n) := (\# \text{ of partitions of } n)$,

$p_k(n) := (\# \text{ of partitions of } n \text{ into } k \text{ parts})$.

Prop. 3.13 (e) yields $\quad p_k(n) = p_k(n-k) + p_{k-1}(n-k)$

$\forall k \geq 1 \quad \forall n \in \mathbb{N}.$

Also, $\quad p(n) = p_0(n) + p_1(n) + \dots + p_n(n).$

<u>Thm. 9.1.</u> $\displaystyle \sum_{n \geq 0} p(n) x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}.$

(The product on the RHS is well-defined, because multiplying a FPS by $\frac{1}{1-x^k}$ does not affect its first k coefficients.)

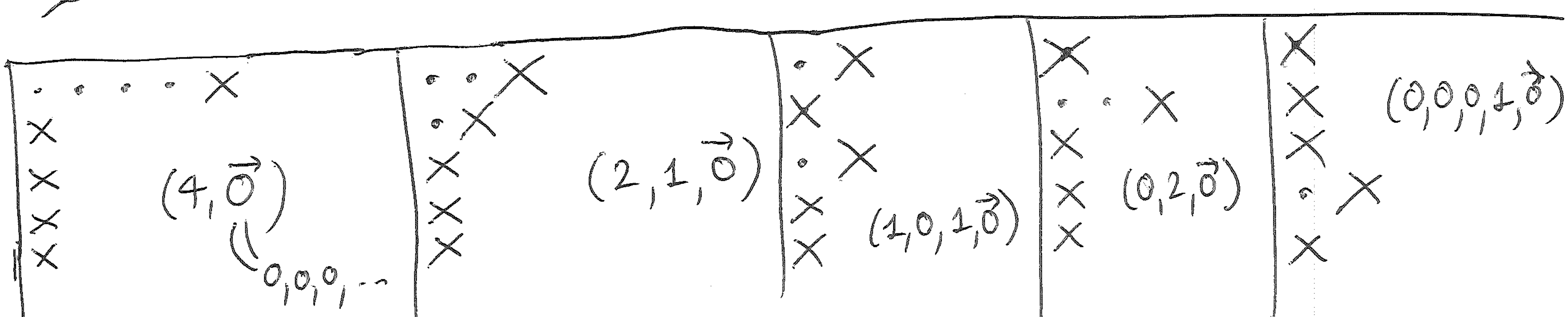
<u>Proof.</u> $\displaystyle \prod_{k=1}^{\infty} \frac{1}{1-x^k} = \prod_{k=1}^{\infty} \left(1 + x^k + (x^k)^2 + (x^k)^3 + \dots\right)$

$\displaystyle = \prod_{k=1}^{\infty} \left(1 + x^k + x^{2k} + x^{3k} + \dots\right)$

$$
\begin{aligned}
&= (1 + x + x^2 + x^3 + x^4 + \cdots) \\
&\quad \cdot (1 + x^2 + x^4 + x^6 + x^8 + \cdots) \\
&\quad \cdot (1 + x^3 + x^6 + x^9 + \cdots) \\
&\quad \cdot (1 + x^4 + x^8 + x^{12} + \cdots) \\
&\quad \cdot (1 + x^5 + x^{10} + x^{15} + \cdots) \\
&\quad \cdot \cdots
\end{aligned}
$$

$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

Ways to get x^4:



What is the coefficient of x^n for a general $n \in \mathbb{N}$?
It is the # of ways to assemble x^n ~~from an add~~ by picking an addend out of each factor.
In other words: It is the # of all $(m_1, m_2, m_3, \ldots) \in \mathbb{N}^\infty$ such that $1m_1 + 2m_2 + 3m_3 + \ldots = n$.

But

$$\{\text{partitions of } n\} \to \left\{ (m_1, m_2, m_3, \ldots) \in \mathbb{N}^\infty \;\middle|\; 1m_1 + 2m_2 + 3m_3 + \ldots = n \right\},$$

$$\lambda \mapsto (\#\text{ of parts } 1 \text{ in } \lambda, \\ \#\text{ of parts } 2 \text{ in } \lambda, \\ \#\text{ of parts } 3 \text{ in } \lambda, \\ \ldots)$$

is a bijection. Thus, our coefficient is the # of partitions of n. But this is $p(n)$. $\square$