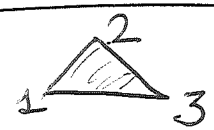
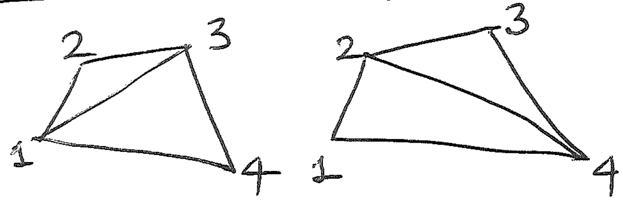
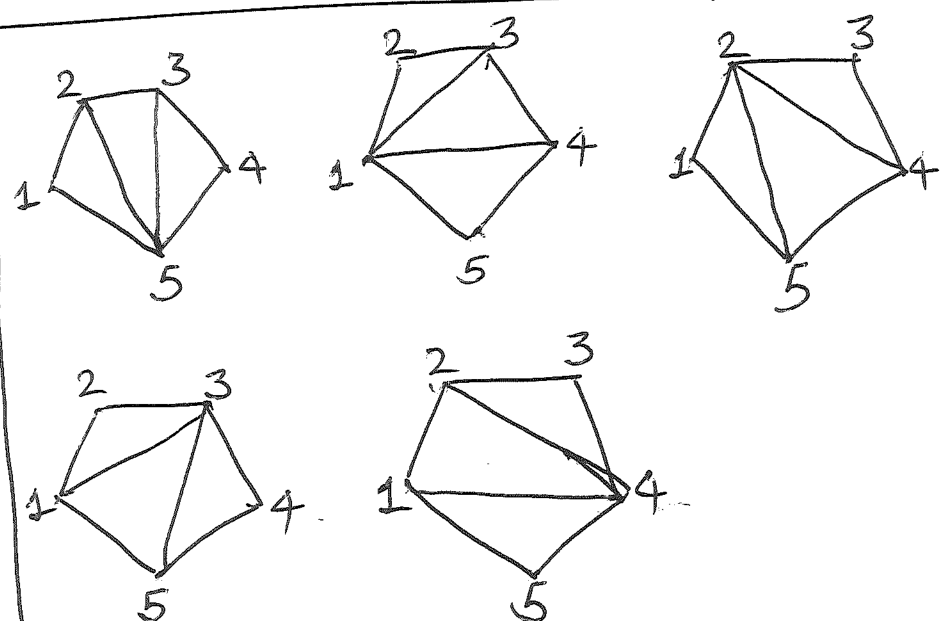
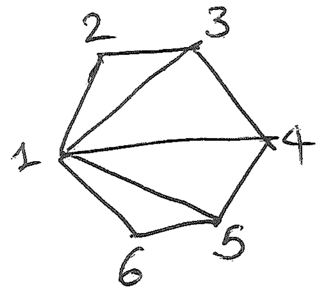
Rmk. Fix an $n \geq 3$, and a convex n-gon $\mathcal{G}_n$.
How many ways are there to triangulate $\mathcal{G}_n$ (i.e., subdivide it into triangles whose vertices are vertices of $\mathcal{G}_n$)? Call this number T_n.

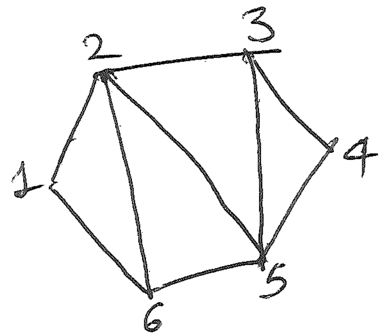
Ex:

$n=3$:		$T_3 = 1$
$n=4$:		$T_4 = 2$
$n=5$:		$T_5 = 5$

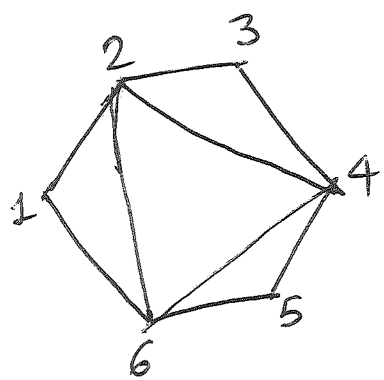
$n=6$:

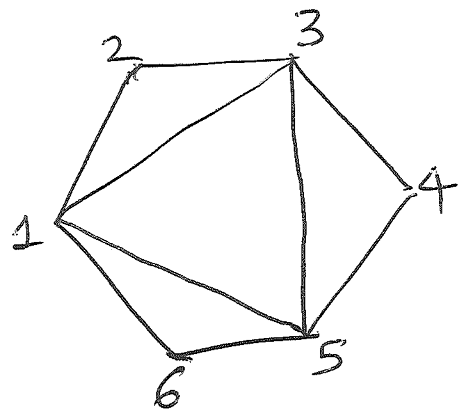


and 5 similar ones



and 5 similar ones



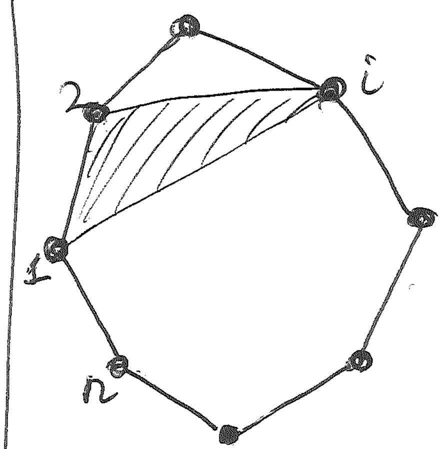


$T_6 = 14$

Answer: $T_n = C_{n-2}$.

Proof sketch. Count triangulations of G_n according to the third vertex $i \in \{3, 4, \dots, n\}$ of the unique triangle that contains the side 12.

Thus,
$$T_n = \sum_{i=3}^{n} T_{i-1} \cdot T_{n-i+2},$$



because after choosing i, we still need to triangulate the $(i-1)$-gon $23\cdots i$ and the $(n-i+2)$-gon $i(i+1)\cdots n1$.

Comparing this with $c_m = \sum_{k=0}^{m-1} c_k c_{m-1-k}$ (Cor. 6.4), we get (by strong induction) $T_n = C_{n-2}$. $\square$

This goes back to Euler.

6.3. Ballot numbers & k-legal paths

Fix a positive integer k.

Def. A LP $\mathbf{v}$ is said to be <u>k-legal</u> if $x \geq ky$ $\forall (x,y) \in \mathbf{v}$.

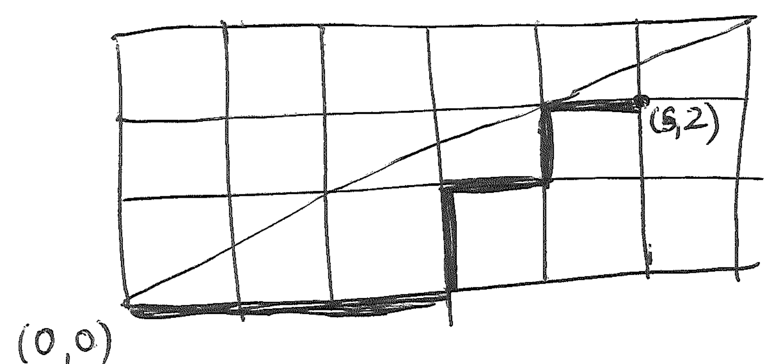
(This means $\mathbf{v}$ never strays above the "flattened diagonal" $x = ky$.)

Given $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, we let

$$L_{n,m,k} := (\# \text{ of } k\text{-legal LPs from } (0,0) \text{ to } (n,m)).$$

(Thus, $L_{n,m,1} = L_{n,m}$.)

Ex: An example for $k=2$, counted into $L_{5,2,2}$:



Lem. 6.5. Let $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

(a) If $n<0$ and $m<0$, then $L_{n,m,k}=0$.

(b) If $n<km$, then $L_{n,m,k}=0$.

(c) $L_{0,0,k}=1$.

(This generalizes Lem. 6.2.)

Prop. 6.6. (a) $L_{n,m,k} = L_{n-1,m,k} + L_{n,m-1,k}$

for any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ satisfying $n \geq km$ and $(n,m) \neq (0,0)$.

(b) $$L_{n,m,k} = \binom{n+m}{m} - k\binom{n+m}{m-1}$$

for any $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $n \geq km-1$.

(c) $$L_{n,m,k} = \frac{n+1-km}{n+1}\binom{n+m}{m}$$

for any $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $n \geq km-1$.

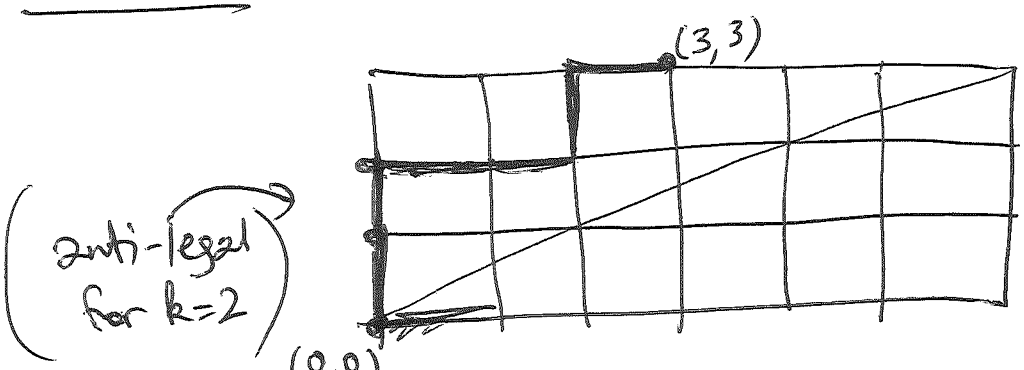
(d) $$L_{km,m,k} = \frac{1}{km+1}\binom{(k+1)m}{m} \quad \forall\, m \in \mathbb{N}.$$

(This generalizes ~~Th~~ Prop. 6.3.)

The numbers $\frac{1}{km+1}\binom{(k+1)m}{m}$ are called <u>Fuss-Catalan numbers</u>.

<u>Proofs of Lem. 6.5 & Prop. 6.6.</u> See HW5. $\square$

<u>Remark.</u> A LP v is called <u>k-anti-legal</u> if $x \leq ky$ $\forall (x,y) \in v$.



Given $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, let

$L'_{n,m,k} :=$ (# of k-anti-legal LPs from $(0,0)$ to (n,m)).

There is no such simple formula for $L'_{n,m,k}$ as Prop. 6.6(b).
But there is an analogue of Prop. 6.6(d), since

$L'_{km,m,k} = L_{km,m,k}$, since the map

$\{k\text{-legal LPs from } (0,0) \text{ to } (km,m)\}$
$\to \{k\text{-anti-legal LPs from } (0,0) \text{ to } (km,m)\}$,

$$v \longmapsto (\text{rotation of } v \text{ by } 180^\circ \text{ around } (\tfrac{km}{2}, \tfrac{m}{2}))$$

is a bijection.

<u>Note</u>: $L_{n,m}$ and occasionally $L_{n,m,k}$ are called the <u>ballot numbers</u>, and the problem of computing them the <u>ballot problem</u>. See [Renault, "Four Proofs of the Ballot Theorem"].

6.4. Rational Catalan numbers

<u>Prop. 6.7.</u> Let a and b be two coprime positive integers. (Recall: a and b are coprime $\Longleftrightarrow$ the fraction a/b is reduced.)

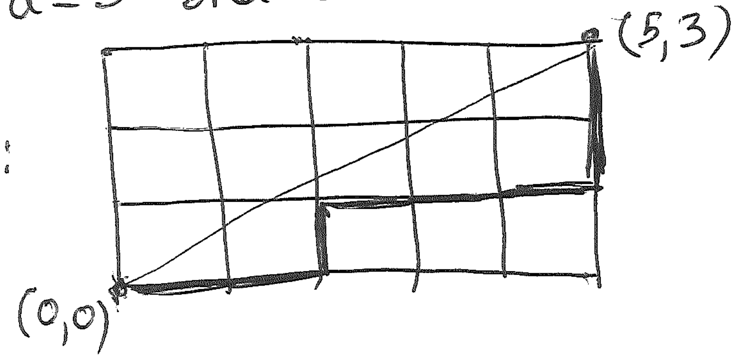
A LP v is said to be <u>(a,b)-legal</u> if $ax \geq by$ $\forall (x,y) \in v$.

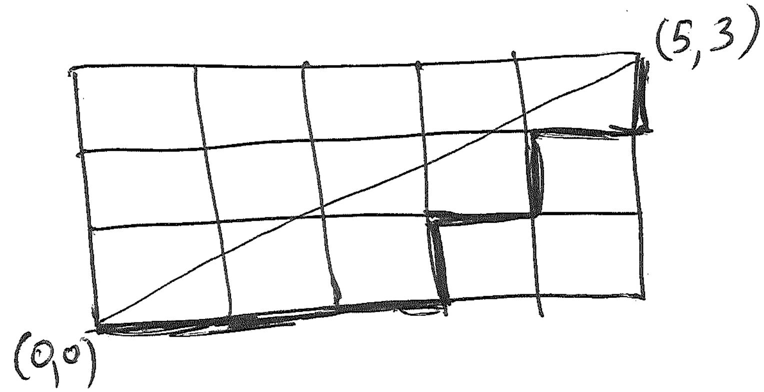
(This means that v never strays above the "slanted diagonal" $ax = by$.)

Then, the # of (a,b)-legal LPs from $(0,0)$ to (b,a) is $\frac{1}{a+b}\binom{a+b}{a}$.

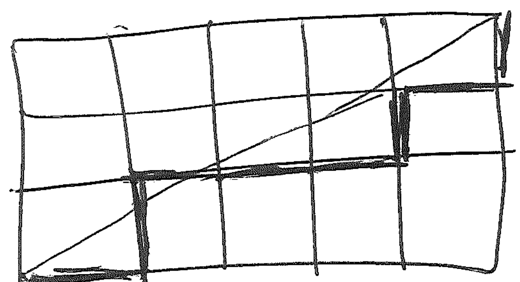
<u>Example:</u> $a=3$ and $b=5$.

(a,b)-legal:





not (a,b)-legal:



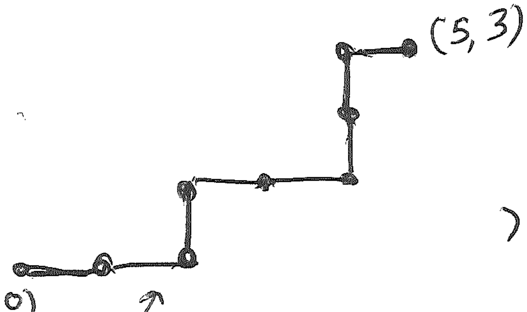
<u>Proof sketch.</u> First, observe that the only points $(u,v)\in\mathbb{Z}^2$ on the $ax=by$ diagonal are of the form (pb, pa) for $p\in\mathbb{Z}$ (since a and b are coprime).

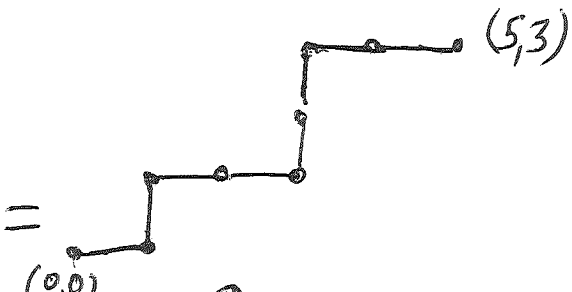
Also, the # of <u>all</u> LPs from $(0,0)$ to (b,a) is $\binom{a+b}{a}$ (by

Prop. 6.1).

For any LP v from $(0,0)$ to (b,a), we define the <u>shift</u> of v to be the LP $S(v)$ from $(0,0)$ to (b,a) constructed as follows: Encode v as a sequence of $a+b$ steps (U's or R's), then rotate this sequence cyclically (so the first step goes to the very end), then decode the resulting sequence again to get $S(v)$.

[<u>Ex</u>: If

$v =$ 

(sequence: RRURRUUR), then $S(v) =$ 

(sequence: RURRUURR)]

A geometric way to define $S(v)$ is as follows: Cut-and-paste the first step of v onto the end of v, and then translate the resulting LP to make it start at $(0,0)$ again.

Thus, we have defined a map

$$S : \{\text{LPs from } (0,0) \text{ to } (b,a)\} \to \{\text{LPs from } (0,0) \text{ to } (b,a)\},$$
$$\gamma \mapsto S(\gamma).$$

This map S satisfies $S^{a+b} = \mathrm{id}$, and thus is bijective, i.e., is a permutation of the finite set $\{\text{LPs from } (0,0) \text{ to } (b,a)\}$. Consider its cycles. (Previously, I would have called them "shift-equivalence classes", but now we know about cycles.)

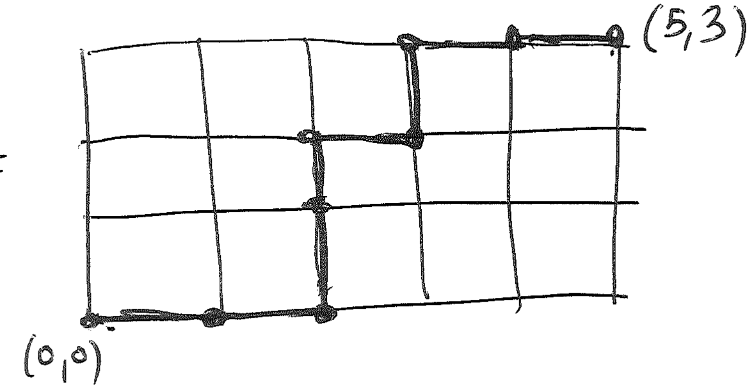
<u>Claim 1:</u> Each cycle of S has size $a+b$.

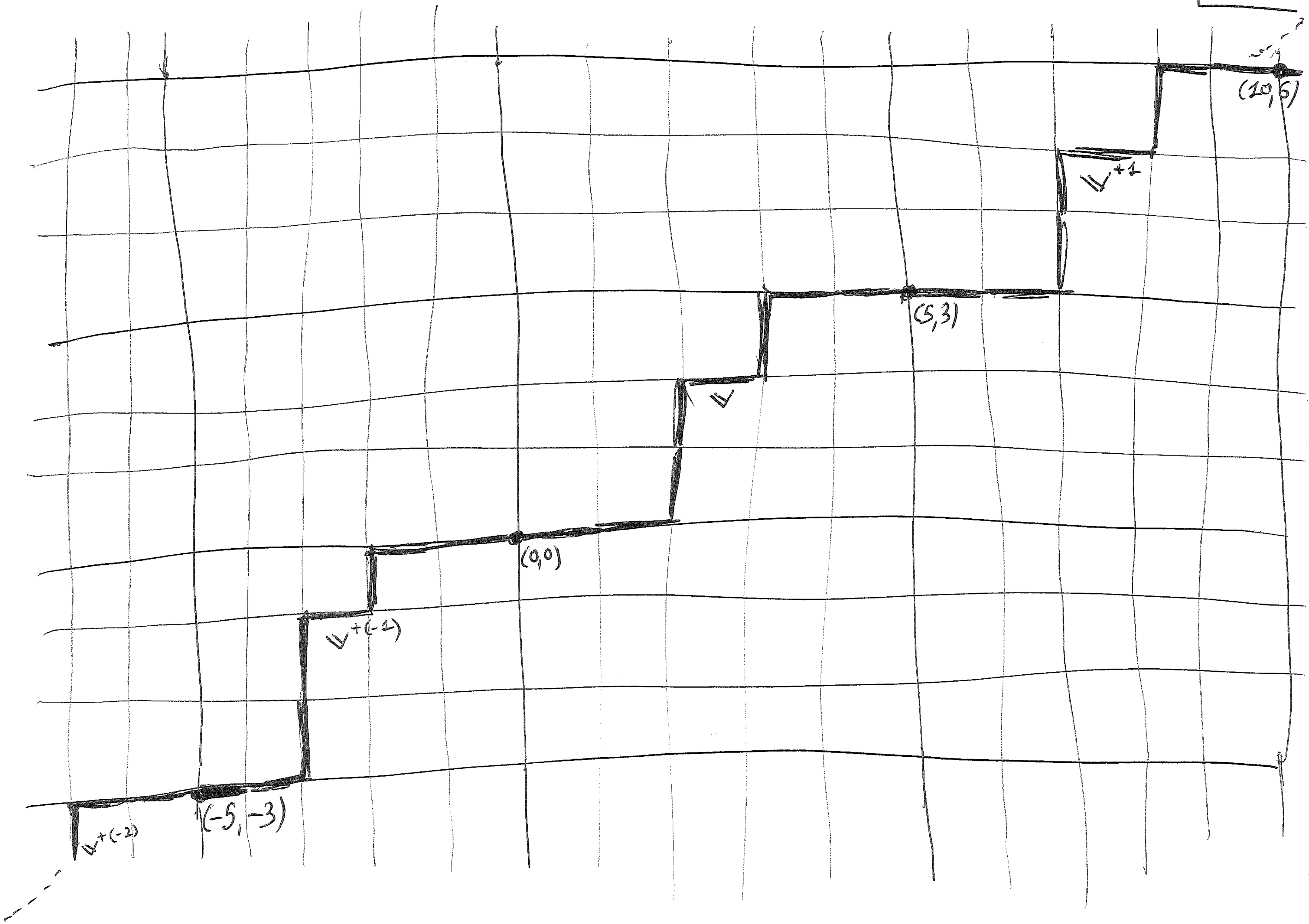
<u>Claim 2:</u> Each cycle of S contains exactly one (a,b)-legal LP.

Once Claims 1 & 2 are proven, it will follow that exactly 1 in $a+b$ LPs from $(0,0)$ to (b,a) is (a,b)-legal, and thus the # of (a,b)-legal LPs is $\frac{1}{a+b}\binom{a+b}{a}$ (since the total # of LPs is $\binom{a+b}{a}$). Hence, it suffices to prove Claims 1 & 2.

To prove them both, let us fix a cycle C of S.
Choose any $v \in C$.
For each $p \in \mathbb{Z}$, let v^{+p} be the LP v translated by (pb, pa).
The LPs v^{+p} for all $p \in \mathbb{Z}$ can be combined to a single "infinite LP" v^{∞} (infinite in both ways), since $v^{+(p-1)}$ ends where v^{+p} starts.

[Ex: $a=3$ and $b=5$:

If $v =$ , then v^{∞} is


(10,6)
U+1
(5,3)
U
(0,0)
U+(-1)
(-5,-3)
U+(-2)

Label the points on $\mathbb{V}^\infty$ by $\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots$

where v_i is the unique point $(x,y) \in \mathbb{V}^\infty$ for which $x+y=i$.

Thus, $v_0 = (0,0)$.

For each $i \in \mathbb{Z}$, let l_i be the line through v_i parallel to the line $ax=by$.

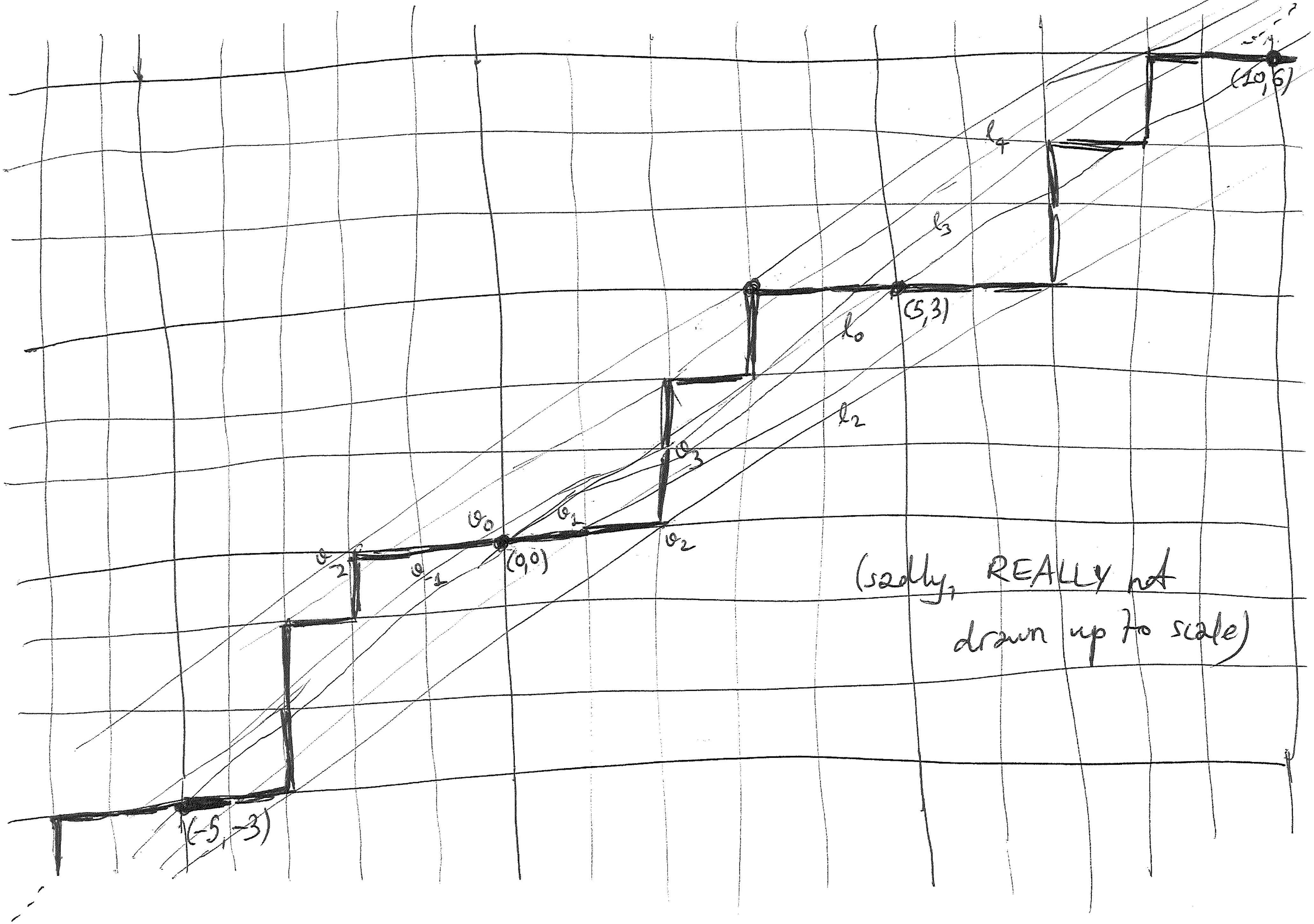
For each $i \in \mathbb{Z}$, we have $v_{i+(a+b)} = v_i + (b,a)$

(since $\mathbb{V}^\infty$ is just $\mathbb{V}$ "repeated over & over")

and thus $l_{i+(a+b)} = l_i$.

Hence, the lines $\ldots, l_{-2}, l_{-1}, l_0, l_1, l_2, \ldots$ repeat with period $a+b$.

Moreover, if we had $l_i = l_j$ for some $0 \le i < j < a+b$, then v_i and v_j would lie on a common line parallel to $ax=by$, which would quickly contradict the assumption that a and b are coprime. Thus, $l_0, l_1, \ldots, l_{a+b-1}$ are distinct.



(sadly, REALLY not drawn up to scale)

We have $\mathbf{v} = (v_0, v_1, \ldots, v_{a+b})$.

~~Hence, $s^k(\mathbf{v}) = (v_k, v_{k+1}, \ldots, v_{k+(a+b)})$ $\forall k \in \mathbb{Z}$ (proof: induction up & down).~~

Hence, for each $k \in \mathbb{Z}$, the LP $s^k(\mathbf{v})$ is just $(v_k, v_{k+1}, \ldots, v_{k+(a+b)})$, translated to start at $(0,0)$.

[Proof of Claim 1: We must prove $|C| = a+b$.

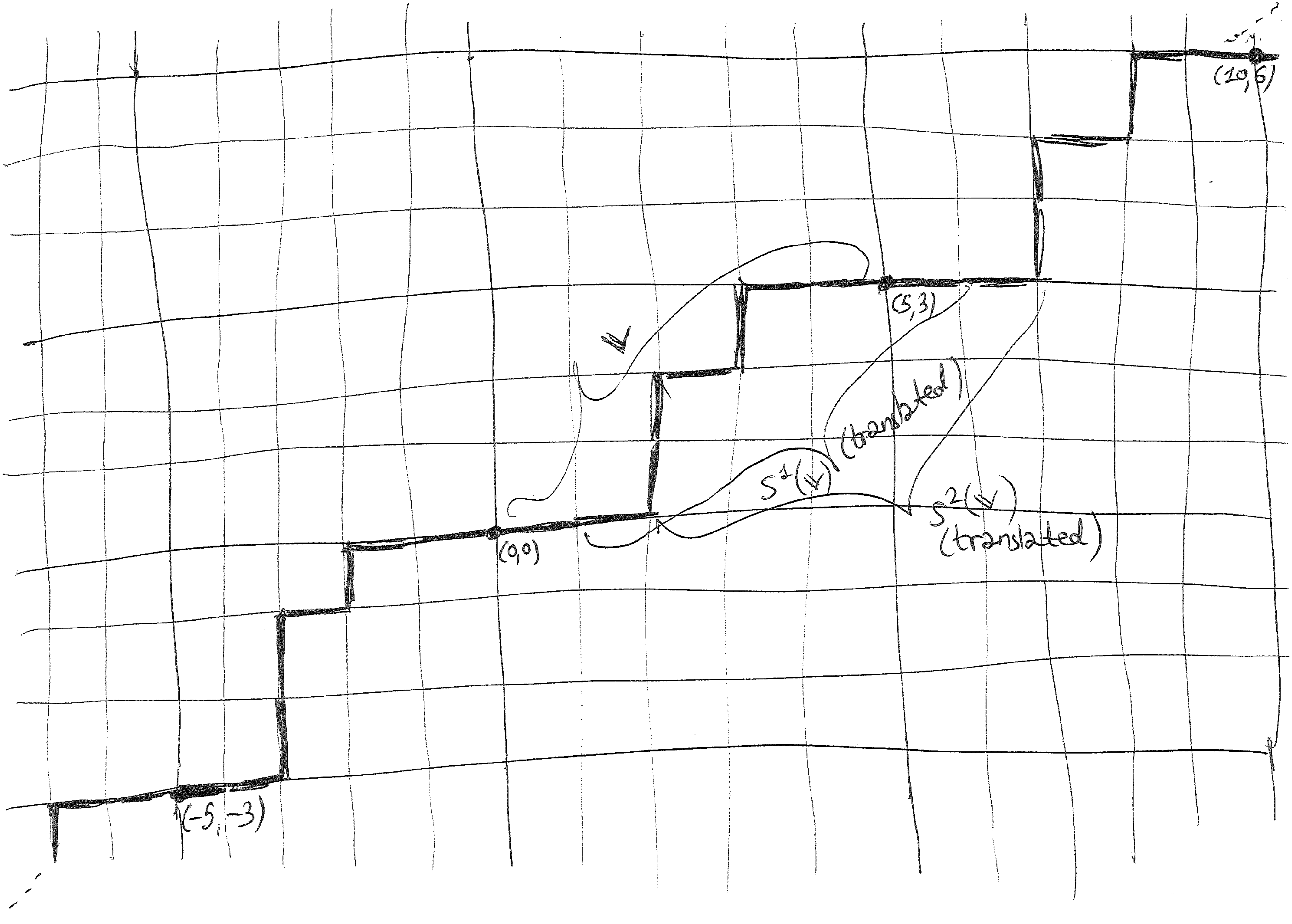
Let $d = |C|$. So we must prove $d = a+b$. Assume the contrary.

Since C is a cycle of S of size d, we have $s^d(\mathbf{v}) = \mathbf{v}$, while $s^0(\mathbf{v}), s^1(\mathbf{v}), \ldots, s^{d-1}(\mathbf{v})$ are distinct.

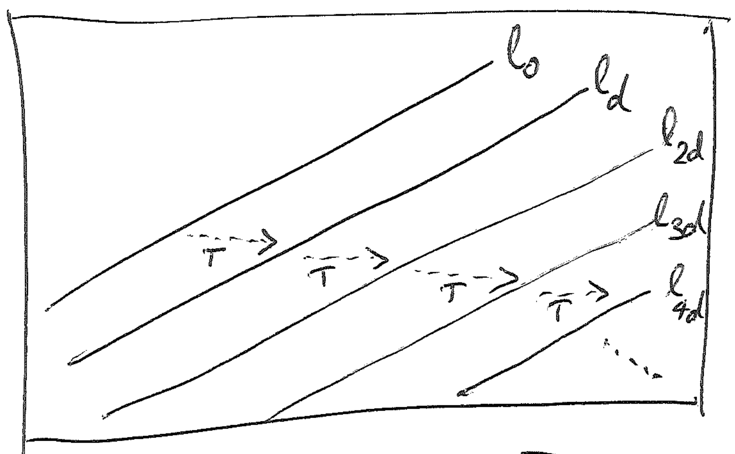
Hence, $d \leq a+b$ (since $s^{a+b}(\mathbf{v}) = \mathbf{v}$).

Thus, $d < a+b$ (since we assumed $d \neq a+b$).

Hence, $\ell_d \neq \ell_0$ (since $\ell_0, \ell_1, \ldots, \ell_{a+b-1}$ are distinct).


(10,6)
(5,3)
(0,0)
(-5,-3)
v
S^1(v)
(translated)
S^2(v)
(translated)

But $s^d(v) = v$. Hence, $v_{i+d} - v_{i+d-1} = v_i - v_{i-1}$ $\forall i \in \mathbb{Z}$.
Hence, the translation T that sends v_0 to v_d must send v_i to v_{i+d} $\forall i \in \mathbb{Z}$.
Thus, this translation T sends l_i to l_{i+d} $\forall i \in \mathbb{Z}$.
$\Rightarrow$ It sends $l_0 \to l_d \to l_{2d} \to l_{3d} \to \cdots$
$\Rightarrow$ Since $l_d \neq l_0$, all the lines $l_0, l_d, l_{2d}, l_{3d}, \ldots$ are distinct.



Thus, there are ∞ many distinct l_i's.
This contradicts the fact that $\ldots, l_{-2}, l_{-1}, l_0, l_1, l_2, \ldots$ repeats with period $a+b$.

So Claim 1 is proven.]

[Proof of Claim 2: For any $i \in \mathbb{Z}$, we have the following chain of equivalences:

$(s^i(\mathbf{v})$ is (a,b)-legal$)$

$\iff$ $((v_i, v_{i+1}, \ldots, v_{i+(a+b)})$ never rises above $\ell_i)$

(by parallel translation)

$\iff$ $(\mathbf{v}^\infty$ never rises above $\ell_i)$

(since $\mathbf{v}^\infty$ just consists of "copies" of $(v_i, v_{i+1}, \ldots, v_{i+(a+b)})$)

$\iff$ (no v_j is above ℓ_i)

$\iff$ (no ℓ_j is northwest of ℓ_i)

$\iff$ (ℓ_i is the northwesternmost of the parallel lines $\ldots, \ell_{-2}, \ell_{-1}, \ell_0, \ell_1, \ell_2, \ldots$)

$\iff$ (ℓ_i is the northwesternmost of the parallel lines $\ell_0, \ell_1, \ldots, \ell_{a+b-1}$).

This happens for exactly one $i \in \{0, 1, \ldots, a+b-1\}$ (since

$l_0, l_1, \ldots, l_{a+b-1}$ are distinct).

Thus, there is exactly one $i \in \{0, 1, \ldots, a+b-1\}$ for which $s^i(v)$ is (a,b)-legal.

Therefore, the cycle C contains exactly one (a,b)-legal LP.

Claim 2 is proven.] $\square$

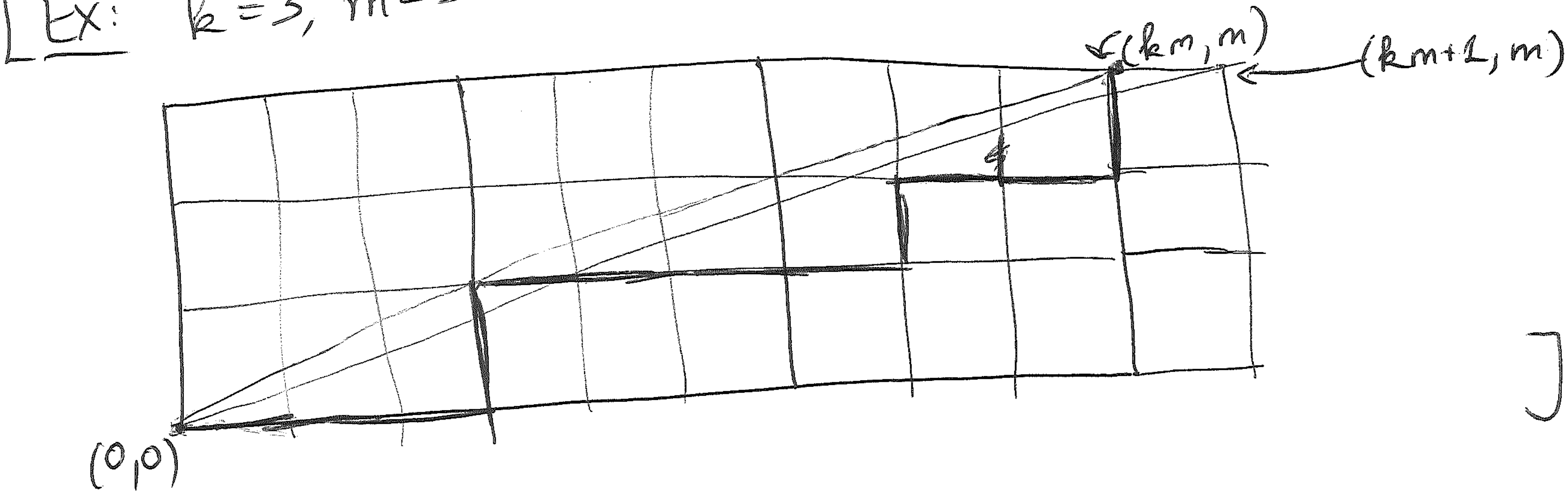
<u>Second proof of Prop. 6.6 (d) (sketched).</u> The positive integers m and $km+1$ are coprime.

Thus, Prop. 6.7 (applied to $a = m$ & $b = km+1$) yields that

(# of $(m, km+1)$-legal LPs from $(0,0)$ to $(km+1, m)$)

$$= \frac{1}{m+km+1}\binom{m+km+1}{m} \underset{\text{(absorption identity)}}{=} \frac{1}{km+1}\binom{(k+1)m}{m}.$$

[Ex: $k=3$, $m=3$



]

Now, the map

$\{k\text{-legal LPs from } (0,0) \text{ to } (km,m)\}$

$\to \{(m, km+1)\text{-legal LPs from } (0,0) \text{ to } (km+1, m)\}$,

$(v_0, v_1, \ldots, v_{km+m}) \longmapsto ((0,0), v_0+(1,0), v_1+(1,0), \ldots, v_{km+m}+(1,0))$

is well-defined & bijective (this is not obvious, but not too hard to show). Hence, the # we've computed is the # we've been looking for. $\square$