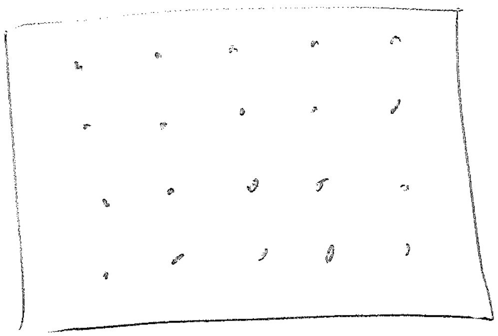
6. Lattice path counting

6.1. Basics

Def. The *integer lattice* is the set $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$.

Its elements are called *points*, and can be added and subtracted entrywise (e.g., $(a,b)-(c,d) = (a-c, b-d)$).



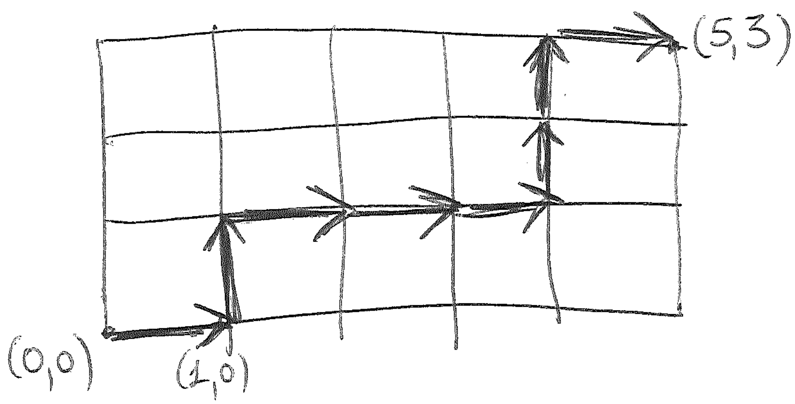
If $(a,b) \in \mathbb{Z}^2$ and $(c,d) \in \mathbb{Z}^2$, then a *lattice path (LP)* from (a,b) to (c,d) means

- (intuitively) a path from (a,b) to (c,d) that uses only 2 kinds of steps:
  - "up-steps" (U) going $(c,d) \mapsto (c, d+1)$;
  - "right-steps" (R) going $(c,d) \mapsto (c+1, d)$.
- (rigorous) a tuple $(v_0, v_1, \ldots, v_n)$ with $v_i \in \mathbb{Z}^2$ and

$v_0 = (a,b)$ and $v_n = (c,d)$ and

$$v_i - v_{i-1} \in \{ \underbrace{(0,1)}_{\text{up-step}}, \underbrace{(1,0)}_{\text{right step}} \} \quad \forall i \in [n].$$

Ex:



is a LP from $(0,0)$ to $(5,3)$.

It is the 9-tuple $((0,0),(1,0),(1,1),(2,1),\dots)$, and can be specified by its "step sequence" RURRRUUR (if $(0,0)$ is known).

Prop. 6.1. Let $(a,b) \in \mathbb{Z}^2$ and $(c,d) \in \mathbb{Z}^2$. Then,

$$(\#\text{ of LPs from } (a,b) \text{ to } (c,d)) = \begin{cases} \binom{c+d-a-b}{c-a}, & \text{if } c+d \geq a+b; \\ 0, & \text{if } c+d < a+b. \end{cases}$$

Proof. In each LP, the x-coordinates of the points weakly increase at each step, and so do the y-coordinates. Thus, LPs from (a,b) to (c,d) can only exist when $c \geq a$ and $d \geq b$.

Furthermore, each step of a LP increases

(x-coordinate) + (y-coordinate)

by exactly 1. Hence, each LP $(v_0, \ldots, v_n)$ from (a,b) to (c,d) has $n = c+d-a-b$. Thus, if $c+d < a+b$, the # of LPs is 0.

~~Now~~ Otherwise, the bijection

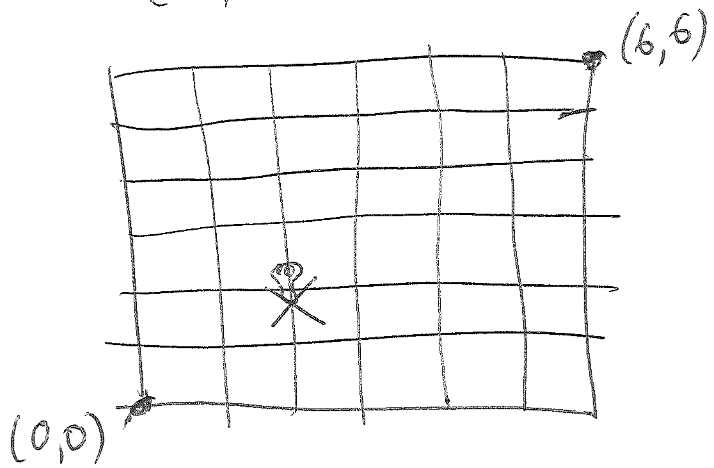
$$\{\text{LPs from } (a,b) \text{ to } (c,d)\} \to \{(c-a)\text{-elt. subsets of } [c+d-a-b]\},$$

$$(v_0, v_1, \ldots, v_n) \mapsto \{i \in [n] \mid v_i - v_{i-1} = (1,0)\}$$

shows that the # of LPs is $\dbinom{c+d-a-b}{c-a}$. $\square$

Def. Let $\mathbb{v} = (v_0, v_1, \ldots, v_n)$ be a LP from (a,b) to (c,d). Let $p \in \mathbb{Z}^2$. We say that $p \in \mathbb{v}$ ("p lies on $\mathbb{v}$") if $p \in \{v_0, v_1, \ldots, v_n\}$.

Exercise: Find the # of LPs $\mathbb{v}$ from $(0,0)$ to $(6,6)$ such that $(2,2) \notin \mathbb{v}$.



Answer: It is
(# of LPs from $(0,0)$ to $(6,6)$)
$-$ (# of LPs from $(0,0)$ to $(2,2)$)
$\cdot$ (# of LPs from $(2,2)$ to $(4,4)$)

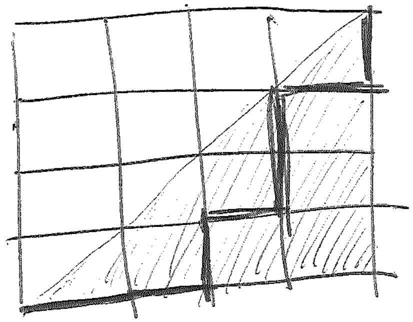
$$\overset{\text{Prop. 6.1}}{=} \binom{6+6}{6} - \binom{2+2}{2} \cdot \binom{4+4}{4}.$$

6.2. Sub-diagonal paths & Catalan numbers

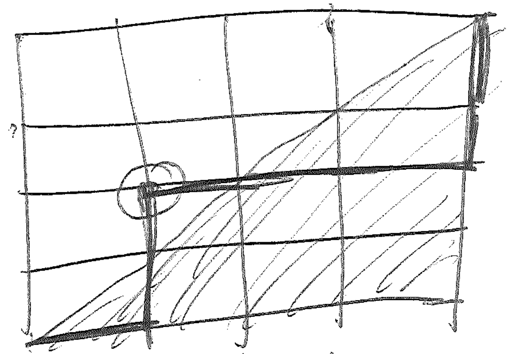
Def. A LP $\mathbb{v}$ is said to be *legal* if $x \geq y$ $\forall (x,y) \in \mathbb{v}$.
(Visually, this means that $\mathbb{v}$ never strays above the diagonal

$x=y$.)

Examples:



legal



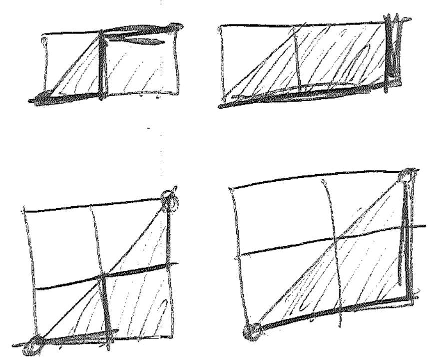
not legal

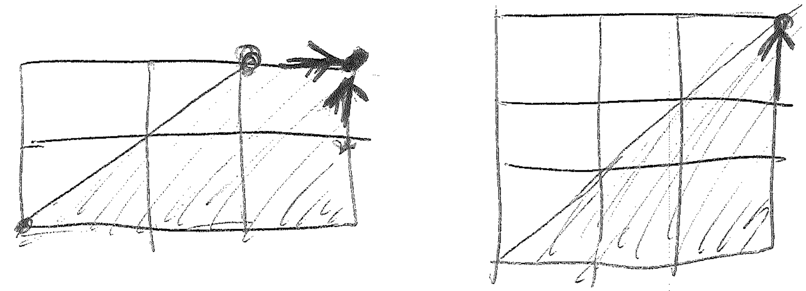
Def. If $n\in\mathbb{Z}$ and $m\in\mathbb{Z}$, then $L_{n,m}$:= (# of legal LPs from (0,0) to (n,m)).

Ex:

$n \backslash m$	0	1	2	3
0	1	0	0	0
1	1	1	0	0
2	1	2	2	0
3	1	3	5	5

Table of $L_{n,m}$





Lemma 6.2. Let $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

(a) If $n < 0$ or $m < 0$, then $L_{n,m} = 0$.

(b) If $n < m$, then $L_{n,m} = 0$.

(c) $L_{0,0} = 1$.

Proof. Basically trivial. (See Spring 2018 Math 4707 mt#2 $\square$ 90.2).

Proposition 6.3. (a) $L_{n,m} = L_{n-1,m} + L_{n,m-1}$ for any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ satisfying $n \geq m$ & $(n,m) \neq (0,0)$.

(b) $L_{n,m} = \binom{n+m}{m} - \binom{n+m}{m-1}$ for any $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $n \geq m-1$.

(c) $L_{n,m} = \dfrac{n+1-m}{n+1} \binom{n+m}{m}$ for any $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $n \geq m-1$.

(d) $L_{n,n} = \frac{1}{n+1}\binom{2n}{n}$ for any $n \in \mathbb{N}$.

<u>Proof.</u> (See [loc. cit.] for details.)

(a) Count LPs to (n,m) according to their last step.

(b) Induction on $n+m$, using (a).

(c) Simple algebra.

(d) Apply (c) to $m=n$. $\square$

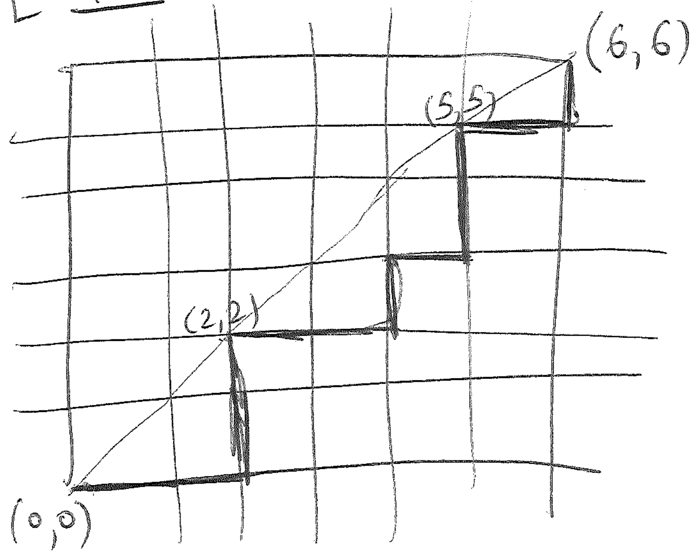
<u>Def.</u> For any $n \in \mathbb{N}$, the number $L_{n,n} = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$ is called the n-th <u>Catalan number</u> and written C_n.

<u>Cor. 6.4.</u> For any $n > 0$, we have $\cancel{C_n = \sum_{k=0}^{n-1} C_k C_{n-k}}$

$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}.$$

<u>Proof.</u> For each legal LP $\mathbb{v} = (v_0, \ldots, v_{2n})$ from $(0,0)$ to (n,n), we let $r(\mathbb{v})$ be the smallest $k \in [n]$ such that $(k,k) \in \mathbb{v}$.

[Ex:



Here, $r(\mathbb{v}) = 2$.]

Now, for each $k \in [n]$, we have

(# of legal LPs $\mathbb{v}$ from $(0,0)$ to (n,n) with $r(\mathbb{v}) = k$)

$$= \underbrace{\left(\text{\# of legal LPs from } (0,0) \text{ to } (k,k) \text{ that do not contain any of } (1,1), (2,2), \ldots, (k-1,k-1)\right)}_{=:\alpha}$$

$$\cdot \underbrace{\left(\text{\# of legal LPs from } (k,k) \text{ to } (n,n)\right)}_{\substack{= (\text{\# of legal LPs from } (0,0) \text{ to } (n-k,n-k)) \\ = L_{n-k,n-k} = C_{n-k}}}$$

$= \alpha \cdot C_{n-k}$,

where α = (# of legal LPs from $(0,0)$ to (k,k) that do not contain any of $(1,1), (2,2), \ldots, (k-1,k-1)$)

= (# of LPs from $(1,0)$ to $(k,k-1)$ that satisfy $x \geq y+1$ for all (x,y) on the path)

= (# of legal LPs from $(0,0)$ to $(k-1,k-1)$) (by parallel translation)

$= L_{k-1,\,k-1} = C_{k-1}$.

Thus, this # is $C_{k-1} \cdot C_{n-k}$.

Summing this over all k, we get $C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}$. $\square$

<u>Proof of a part of Thm. 4.21.</u> We want to prove that

(# of 132-avoiding perms $\sigma \in S_n$) $= C_n$.

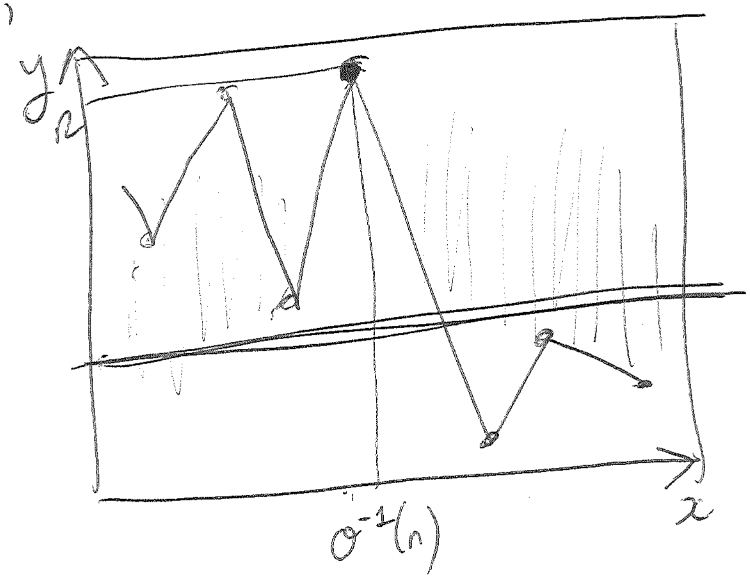
Let $a_n :=$ ~~ℓ~~(# of 132-avoiding perms. $\sigma \in S_n$) $\forall n \in \mathbb{N}$.

Our goal is to prove

$$(61) \qquad a_n = \sum_{k=1}^{n} a_{k-1} a_{n-k} \qquad \forall n > 0,$$

because then the claim will follow by strong induction on n using Cor. 6.4. So let $n > 0$.

If $\sigma \in S_n$, we ~~let~~ consider the plot of σ:



In the one-line notation of σ, each entry left of n is larger than each entry right of n (since otherwise, we would get a 132-pattern).

Thus, each 132-avoiding perm $\sigma \in S_n$ can be constructed as follows:

- Choose $k = \sigma^{-1}(n)$.
- Choose the $k-1$ entries left of n.
  These must be the $k-1$ numbers $n-k, \dots, n-1$, placed in a 132-avoiding order.
  So there are a_{k-1} choices here.
- Choose the $n-k$ entries right of n.
  Here you have a_{n-k} choices here. $\square$

<u>Remark:</u> Fix $n \in \mathbb{N}$. Consider "words" made of n opening parentheses & n closing parentheses.
E.g., for $n=5$, it could be $))()()(()($.
This word is said to be <u>legal</u> if the parentheses can be matched (each opening parenthesis to a closing

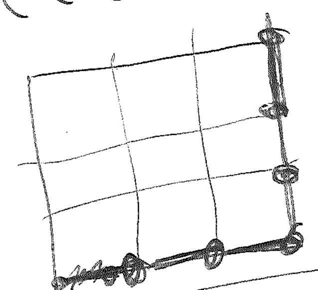
parenthesis to its right). E.g.:

(( ) ( ) (( ))) ( )

How many legal words with n opening & n closing parentheses are there?

[E.g., for $n=3$:

((( ))) ,



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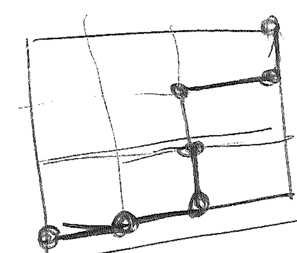

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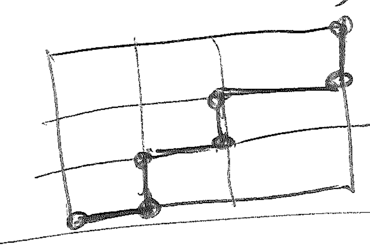

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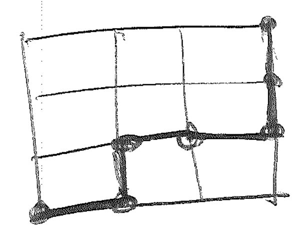

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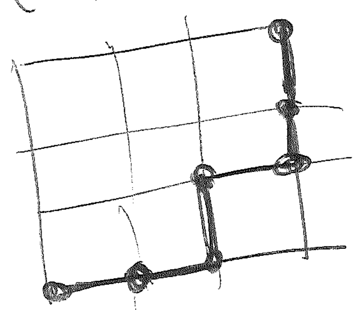

]

The answer is C_n.

Rmk: How many ways are there to "fully" parenthesize a sum of 5 numbers?

$(a + (b + (c + d))) + e$

$(a + (b + c)) + (d + e)$

$(((a + b) + c) + d) + e$

$a + (b + (c + (d + e)))$

$(a + ((b + c) + d)) + e$

$a + (((b + c) + d) + e)$

$(a + b) + ((c + d) + e)$

Call this p_5.

$$p_5 = p_1 p_4 + p_2 p_3 + p_3 p_2 + p_4 p_1.$$

Compare with Cor. 6.4:

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \dots + C_{n-1} C_0.$$

$$\Rightarrow p_n = C_{n-1}.$$

$$\Rightarrow p_5 = C_4 = 14.$$

See [Stanley, "Catalan numbers"] for more examples of C_n as answers for questions.