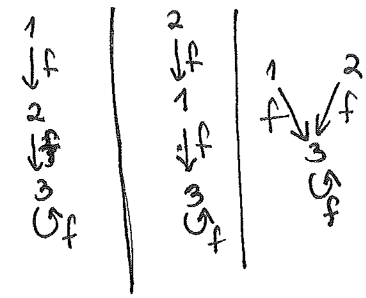
Danj ended with this <u>Thm 5.4</u>: If $b_i := \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}$, then

$$\det \begin{bmatrix} b_1 & -a_{12} & -a_{13} & \cdots & -a_{1,n-1} \\ -a_{21} & b_2 & -a_{23} & & -a_{2,n-1} \\ -a_{31} & & \ddots & & \\ \vdots & & & & \\ -a_{n-1,1} & -a_{n-1,2} & & & b_n \end{bmatrix} = \sum_{\substack{f:[n]\to[n] \\ \{1,\dots,n\} \\ f^n([n])=\{n\}}} \prod_{i=1}^{n-1} a_{i,f(i)}$$

e.g. $n=3$

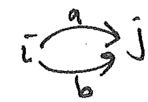
$$\det \begin{array}{c c} & \begin{array}{cc} (1) & (2) \end{array} \\ \begin{array}{c} (1) \\ (2) \end{array} & \begin{bmatrix} a_{12}+a_{13} & -a_{12} \\ -a_{21} & a_{21}+a_{23} \end{bmatrix} \end{array} = (a_{12}+a_{13})(a_{21}+a_{23}) - a_{12}a_{21}$$

$$= \cancel{a_{12}a_{21}} + a_{12}a_{23} + a_{13}a_{21} + a_{13}a_{23} - \cancel{a_{12}a_{21}}$$



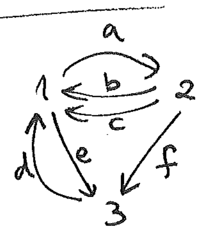
Let's prove it by proving something seeming more general (but not really)...

<u>DEF'N:</u> Given a <u>directed graph</u> $D = (V, A)$, with vertices $V = \{1,2,\dots,n\}$ and directed arcs A

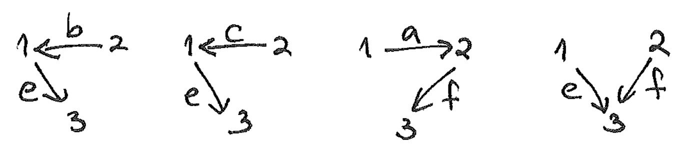


an <u>arborescence α in D directed toward n</u> is a subset $\alpha \subset A$ having exactly one directed path to n for every vertex $i \neq n$

<u>EXAMPLE</u>: $D=$



has 4 arborescences α directed toward $\overset{n=}{3}$:



$\alpha = \{b,e\} \quad \{c,e\} \quad \{a,f\} \quad \{e,f\}$

THM (Directed Matrix-Tree Theorem) In a directed graph $D = (V, A)$, $V = [n]$,

$$\sum_{\substack{\text{arborescences} \\ \alpha \text{ in } D \\ \text{toward } n}} \prod_{\text{arcs } a \in \alpha} a = \det(L)$$

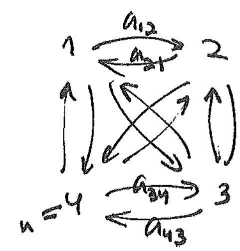
where $L_{ij} = \begin{cases} -\sum\limits_{\text{arcs } i \xrightarrow{a} j} a & \text{if } i \neq j \\ +\sum\limits_{\substack{\text{arcs } i \xrightarrow{a} k \\ k \neq i}} a & \text{if } i = j \end{cases}$ $\quad i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, n-1$

EXAMPLE: For D as before

$$\det \begin{array}{c} \\ ① \\ ② \end{array}\!\!\begin{array}{c} \begin{array}{cc} ① & ② \end{array} \\ \begin{bmatrix} a+e & -a \\ -(b+c) & b+c+f \end{bmatrix} \end{array} = (a+e)(b+c+f) - a(b+c)$$

$$= \cancel{ab} + \cancel{ac} + \underbrace{af + be + ce + ef}_{\text{the 4 arborescences } \alpha \text{ shown earlier!}} - \cancel{ab} - \cancel{ac}$$

NOTE: One recovers Cayley's theorem from the digraph



which is complete bidirected on $\{1, 2, \ldots, n\}$

proof of THM: Induction on # arcs i.e. $\#A$. BASE CASE $\#A = 0$ is easy!

In the inductive step, two cases:

CASE 1: $\nexists$ arcs $i \xrightarrow{f} n$ entering n.

Then LHS $= 0$ since there are no arborescences α toward n in D,

but also $\det(L) = 0$ because every row of L sums to 0,

so $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \text{nullspace}(L)$.

CASE 2: $\exists$ at least one arc $i \xrightarrow{f} n$ entering n.

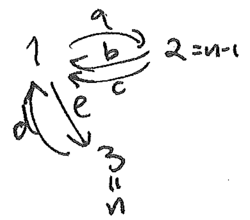
Relabeling vertices, one can assume $i = n-1$, so $n-1 \xrightarrow{f} n$.

(3) Define two new directed graphs

$$D \setminus f = (V,\ A \setminus \{f\}) \qquad \text{and} \qquad D/f = \left(V/_{\{n-1 = n\}},\ A/f\right)$$

$D \setminus f$: deletion of f in D $\qquad$ D/f: contraction of f in D

e.g.



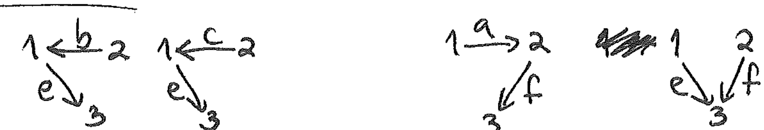
e.g.

and then it's easy to check that if $\mathrm{LHS}(D) := \sum_{\alpha \text{ for } D} \prod_{a\in\alpha} a$ = left side of THM,

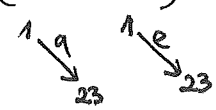
$$\mathrm{LHS}(D) = \sum_{\alpha} \prod_{a\in\alpha} a = \sum_{f\notin\alpha} \prod_{a\in\alpha} a + \sum_{f\in\alpha} \prod_{a\in\alpha} a$$

$$= \mathrm{LHS}(D\setminus f) + f\,\mathrm{LHS}(D/f)$$

e.g.



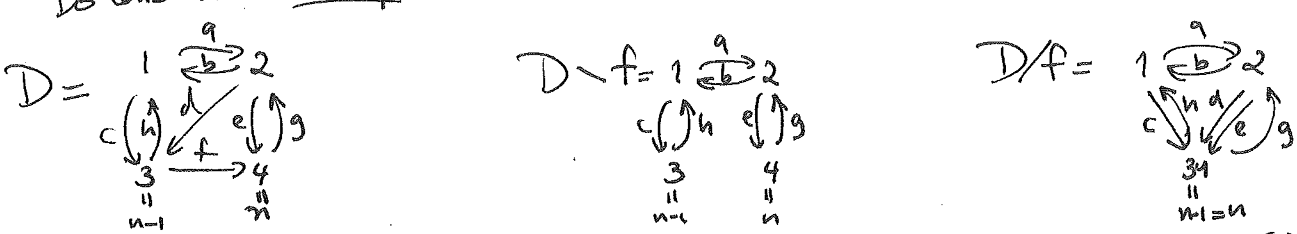
$$(be + ce) + \underbrace{(af + ef)}_{f(a+e)}$$



So it only remains to show

$$\det L(D) = \det L(D\setminus f) + f \cdot \det L(D/f)$$

Do this via example ...



$$\det \begin{array}{c} \\ ① \\ ② \\ ③ \end{array}\!\!\begin{array}{ccc} ① & ② & ③ \\ \end{array}\begin{bmatrix} a+c & -a & a+c \\ -b & b+d+e & -d \\ \boxed{-h \quad 0} & & h+f \end{bmatrix} \overset{?}{=} \det \begin{bmatrix} a+c & -a & a+c \\ -b & b+d+e & -d \\ \boxed{-h \quad 0} & & h \end{bmatrix} + \underbrace{f \cdot \det \begin{bmatrix} a+c & -a \\ -b & b+d+e \end{bmatrix}}_{= \det \begin{bmatrix} a+c & -a & 0 \\ -b & b+d+e & 0 \\ \boxed{-h \quad 0} & & f \end{bmatrix}}$$

replicate these here

This now follows from expansion along last column. ∎

(4)

COROLLARY (Undirected/Kirchhoff's) Matrix-Tree Thm 1847

An (undirected) graph $G = (\underset{\{1,2,\ldots,n\}}{\underset{\shortparallel}{V}}, E)$ has # of spanning trees equal to $\det(L)$ where
$$\underset{\substack{i=1,\ldots,n-1\\ j=1,\ldots,n-1}}{L_{ij}} = \begin{cases} +\deg(i) & \text{if } i=j \\ -(\#\text{edges from } i \text{ to } j) & \text{if } i\neq j \end{cases}$$

proof:

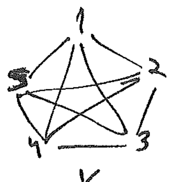
G gives rise to digraph D via $i \overset{e}{\text{—}} j \rightsquigarrow i \rightleftarrows j$

with #spanning trees in G = # arborescences toward n in D.

Apply previous result ∎

EXAMPLE: #spanning trees in K_n = complete graph on $\{1,2,\ldots,n\}$

$n=5$



K_5

$$= \det L(K_n)$$
$$= \det \begin{array}{c|c} & \begin{matrix} (1) & (2) & \cdots & (n-1) \end{matrix} \\ \begin{matrix} (1) \\ (2) \\ \vdots \\ (n-1) \end{matrix} & \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & & -1 \\ -1 & & \ddots & -1 \\ -1 & -1 & \cdots -1 & n-1 \end{bmatrix} \end{array}$$

$$= \underset{\substack{\text{identity}\\ \text{matrix}\\ \text{of size } n-1}}{n I_{n-1}} - \underset{\substack{\text{all ones}\\ \text{matrix}\\ \text{of size } n-1}}{J_{n-1}} \quad \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & & \vdots \\ 1 & \cdots & & 1 \end{bmatrix}$$

One way to quickly evaluate $\det L(K_n)$ is to use the fact that it should be the product $\lambda_1\lambda_2\cdots\lambda_{n-1}$ where $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ are the eigenvalues of $L(K_n)$.

Eigenvectors & eigenvalues for J_{n-1} are easy: $J_{n-1}\begin{bmatrix}1\\1\\ \vdots\\1\end{bmatrix} = \begin{bmatrix}n-1\\n-1\\ \vdots\\n-1\end{bmatrix} = (n-1)\begin{bmatrix}1\\1\\ \vdots\\1\end{bmatrix}$ ← eigenvalue $n-1$

and any $n-2$ vectors $v_1, v_2, \ldots, v_{n-2}$ that give a basis for the perp space $\begin{bmatrix}1\\ \vdots\\1\end{bmatrix}^\perp$ will have $Jv_i = \underline{0} = 0 \cdot v_i$ ← eigenvalues $0,0,\ldots,0$

So J_{n-1} has eigenvalues $(n-1, \overbrace{0,0,\ldots,0}^{n-2 \text{ times}})$

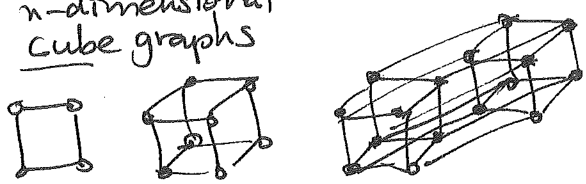
and then $L(K_n) = nI_{n-1} - J_{n-1}$ has the same eigenvectors with eigenvalues $(\lambda_1, \ldots, \lambda_{n-1}) = (n-(n-1), n-0, n-0, \ldots, n-0) = (1, \underbrace{n, n, \ldots, n}_{n-2 \text{ times}})$.

Therefore # spanning trees in $K_n = \lambda_1\lambda_2\cdots\lambda_{n-1}$
$= 1\cdot\underbrace{n\cdot n\cdots n}_{n-2} = n^{n-2}$,

another proof of Cayley's formula.

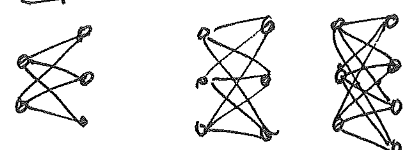
REMARKS:

① There are other families of graphs G where one can count spanning trees ~~using~~ by evaluating $\det L(G)$ via eigenvalues, e.g. n-dimensional cube graphs



$n=2$ $n=3$ $n=4$

complete bipartite graphs $K_{m,n}$



$K_{2,3}$ $K_{3,3}$ $K_{3,4}$ ...

② $\det L(G)$ gives a very fast ($\leq C\cdot n^3$ steps) algorithm to compute # spanning trees in G, using Gaussian elimination; every elimination step changes det in a predictable way.